

MICROPOLAR MATERIALS

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Eringen's theory of micropolar continua is summarized for materials scientists and engineers and reviewed in its state-of-the-art. Kinematic measures are introduced for solids and fluids (micropolar strain and strain-rate tensors). Balance equations are presented, including the non-classical balance law for the microinertia tensor. Non-linear and linear constitutive equations (material models) are given for anisotropic and isotropic solids and fluids. The physical significance of the material parameters is explained and the reduction of their number due to symmetry discussed. With examples from literature, it is shown that micropolar theory can be a useful framework for modeling solid composites, porous media, and suspensions.

INTRODUCTION

The theory of micropolar continua, introduced in the mid 1960es by A.C. Eringen [1-5], is a non-classical field theory for materials (solids and fluids) with a certain kind of microstructure. In classical continuum mechanics a material particle is assigned a certain position, irrespective of orientation, within a material body at a certain time instant. Micropolar material particles can additionally be oriented. In other words, to each micropolar material particle an additional object is assigned, called "director", which defines the orientation of the material particle. In the theory of micropolar continua, contrary to other microcontinuum field theories (Eringen's microstretch and micromorphic continua), this director is rigid, i.e. is used to describe only the microrotation of the material particles. The concept of micropolar materials, based on Noll's concept of simple materials used in rational mechanics [4] and originally designed for viscous fluids [3], has early been extended to elastic solids as well [5]. Eringen himself applied his theory to liquid crystals [2,6,10] and to suspensions with rigid anisometric particles, especially fiber suspensions [2,7-9]. A recent account of the mathematical details of micropolar fluid theory and some of its applications has been given by Lukaszewicz [11], cf. also [12]. This book covers stationary problems (such as the basic Dirichlet problem, flows with heat conduction, flows with diffusion), non-stationary problems (such as non-homogeneous flows, heat-conducting flows) and the behavior of micropolar fluids in problems involving lubrication and porous media. Since it is written from a very strict mathematical point of view (and in a correspondingly abstract style) it will be hardly accessible to materials scientists and engineers who want to obtain

information on the possible applicability of micropolar theory to a practical problem in question. This is the purpose of the present review paper. This paper, based essentially on Eringen's original work, should serve as a practical introduction to the theory of micropolar continua for a broader audience of materials scientists and engineers. Concomitantly with this purpose, it has been attempted to reduce formal details drastically and to keep the number of variables introduced as small as possible. Advantages and achievements as well as shortcomings and unsolved problems are discussed. In two general sections the theoretical framework is introduced, i.e. the dependent and independent variables and the balance equations, valid for both solids and fluids. In the two following sections, material models (constitutive equations) are presented for solids and fluids, respectively. The last section deals with applications of the theory and gives a guide to the relevant literature in which micropolar theory has been invoked for the interpretation of experiments. In this paper we are focusing on the mechanical theory. Note that in particular the treatment of liquid crystals, one of the main applications of micropolar theory so far, requires electrodynamics and is therefore beyond the scope of this paper.

PRELIMINARIES

Micropolar continua can be viewed as a special case of microstretch continua, which again can be viewed as a special case of micromorphic continua [1,2]. All these microcontinuum field theories were introduced by A. C. Eringen and differ from classical (linear and non-linear) field theories by the fact that each material particle is endowed with additional

degrees of freedom. That means, in contrast to classical continuum mechanics (rational mechanics), where the motion (macromotion) of a material particle is fully described by a vector function called deformation function (cf. e.g. one of the standard texts [13-15] or [16]), micromorphic material particles undergo an additional micromotion, corresponding to the rotation and deformation of the material particle at the microscale. Following Eringen, this micromotion can be described by a two-index variable (i.e. an object similar to a tensor), called "director" and denoted χ_{kk} . The indices of the director refer to the actual and referential configuration, respectively, of the material body.¹

In the most general case (micromorphic continua) this director has nine independent components, corresponding to nine additional degrees of freedom (three for microrotation and six for microdeformation). In principle, *micromorphic continua* are almost universal but, due to its complexity, the practical usefulness of this theory is inversely proportional to its generality. A slightly more special case is that of microstretch continua. Here the directors are orthogonal, but permit isotropic expansion or contraction (what Eringen calls a "breathing micromotion") in addition to rotation. Shearing motions (i.e. shear deformations related to a change of shape of the material particles) are not allowed in *microstretch continua*. That means, particles of microstretch continua have four additional degrees of freedom over those of classical continua. Microstretch theory might be adequate to model e.g. bubbly liquids. Even more special is the theory of *micropolar continua*, in which the directors are orthonormal and rigid, i.e.

$$\chi_{kk}\chi_{kk} = \delta_{kl}, \quad (1)$$

where δ_{kl} is the Kronecker delta. That means, only microrotations of material particles (described by three degrees of freedom) are allowed in micropolar continua in addition to the motion at macroscale. Note that the macromotion is given by the usual (vectorial, i.e. one-index) deformation function χ_k , well known in classical continuum mechanics (rational mechanics), cf. [13-16]. In micropolar theory the director may be envisaged as an orthogonal tripod circumscribed by a unit sphere, centered within an anisometric particle, e.g. a triaxial ellipsoid. For rotationally symmetric material particles, e.g. spheroids (biaxial ellipsoids) or cylinders, one-index quantities, e.g. Ericksen's orientational vectors [17] would suffice to determine the orientation.

With the deformation function χ_k and the director χ_{kk} we have the determinants

$$\det \chi_{k,K} > 0 \quad (2)$$

and

$$\det \chi_{kk} > 0, \quad (3)$$

respectively, where the referential gradient of the deformation function $\chi_{k,K}$ is the usual deformation gradient (a second-order tensor)

$$\chi_{k,K} \equiv \frac{\partial \chi_k(X_K, t)}{\partial X_K}. \quad (4)$$

Note that indices after a comma denote partial derivatives. The quantity $\det \chi_{kk}$ represents microvolume changes during microdeformation. For micropolar continua (in contrast to microstretch and micromorphic continua) $\det \chi_{kk} = 1$, i.e. the microvolume does not change with microdeformation, which is a direct consequence of the fact that the directors are rigid, cf. Equation (1). This is in complete analogy with the fact that for $\det \chi_{k,K} = 1$ the (macro-)volume of a body in the actual configuration is the same as in the referential configuration.

The total motion of a micropolar body consists of the macromotion, described by the usual deformation gradients

$$\chi_{k,K}, \quad (5)$$

the *micromotions* (for micropolar bodies only rigid-body microrotation, i.e. rotation of the micropolar material particles), described by the directors

$$\chi_{kk} \quad (6)$$

and their gradients

$$\chi_{kk,L}. \quad (7)$$

Convenient *deformation tensors (strain measures)* for micropolar continua are the Cosserat tensor C_{KL} and the wryness tensor W_{KL} , in Lagrangian (material) representation defined as ([1], p. 16 and [2], p. 7)

$$C_{KL} \equiv \chi_{k,K}\chi_{k,L} \quad (8)$$

and

$$W_{KL} \equiv \frac{1}{2} \varepsilon_{KMN} \chi_{kM,L} \chi_{kN}, \quad (9)$$

respectively, where ε_{KMN} is the Levi-Civita symbol (third-order permutation pseudotensor). These deformation tensors (or their Eulerian, i.e. spatial, equivalents) are fundamental to the construction of constitutive equations of micropolar elastic solids.

Convenient *deformation-rate tensors (strain-rate measures)* for micropolar continua are the second-order tensors a_{kl} and b_{kl} , in Eulerian (spatial) representation defined as ([1], p. 28 and [2], p. 8)

$$a_{kl} \equiv V_{l,k} + \varepsilon_{lkm} v_m \quad (10)$$

and

$$b_{kl} \equiv v_{k,l}, \quad (11)$$

where $V_{l,k}$ is the velocity gradient, v_m the so-called *microrotation vector* and $v_{k,l}$ its gradient. These deforma-

tion-rate tensors are fundamental to the construction of constitutive equations of micropolar viscous fluids. For classical viscous fluids $b_{kl} = 0$ and only the symmetric part of a_{kl} , i.e. the stretching or deformation-rate tensor [18], is relevant. For compatibility conditions and questions of objectivity the reader should consult Eringen's books [1,2].

Since for micropolar continua (in contrast to micro-morphic and microstretch continua) the directors are rigid, it is possible to represent their micromotion as a rigid body rotation (with an angle of rotation ϕ) with respect to an axis n_k . According to Eringen's theorem ([1], p. 9 and [2], p. 7) a (finite) *microrotation tensor* χ_{kl} can be defined which is connected to the *angle of rotation* ϕ via the relation

$$\chi_{kl} = \chi_{kk} \delta_{kl} = \cos\phi \delta_{kl} - \sin\phi \epsilon_{klm} n_m + (1 - \cos\phi) n_k n_l, \quad (12)$$

where δ_{kl} , called "shifters" by Eringen, denote the cosine directors of spatial and material frames. In the case when these frames are chosen to be the same, the δ_{kl} become Kronecker deltas. Similarly, the *microgyration vector* v_k is related to the *rate of rotation* $\dot{\phi}$ by ([1], p. 28 and [2], p. 8)

$$v_k = \dot{\phi} n_k + \sin\phi \dot{n}_k + (1 - \cos\phi)(\mathbf{n} \times \dot{\mathbf{n}})_k. \quad (13)$$

For very small angles ϕ , i.e. within the linear approximation ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$), the microgyration vector v_k is approximately equal to the rate-of-rotation vector (rotational velocity) $\dot{\phi}_k = \widehat{\dot{\phi} n}_k = \dot{\phi} n_k + \phi \dot{n}_k$, i.e.

$$v_k \approx \dot{\phi}_k. \quad (14)$$

BALANCE LAWS OF MICROPOLAR THEORY

The *balance laws applying to micropolar continua* are:

1. Conservation of mass:

$$\dot{\rho} + \rho V_{k,k} = 0 \quad (15)$$

2. Conservation of microinertia:

$$\dot{I}_{kl} + (\epsilon_{kpr} I_{lp} + \epsilon_{lpr} I_{kp}) v_r = 0 \quad (16)$$

3. Balance of linear momentum:

$$T_{kl,k} + \rho (f_l - \dot{V}_l) = 0 \quad (17)$$

4. Balance of angular momentum:

$$M_{kl,k} + \epsilon_{lmn} T_{mn} + \rho (l_l - \dot{\sigma}_l) = 0 \quad (18)$$

5. Balance of energy (first law of thermodynamics):

$$\rho \dot{u} + T_{kl} a_{kl} + M_{kl} b_{lk} + q_{k,k} + \rho Q = 0 \quad (19)$$

6. Entropy inequality (second law of thermodynamics):

$$\rho s - \frac{1}{\theta} q_{k,k} - \frac{1}{\theta} \rho Q \geq 0 \quad (20)$$

In these equations a superimposed dot denotes the material time derivative as before, ρ is the mass density, V_k the velocity vector, I_{kl} the *microinertia tensor*, v_k the microgyration vector, T_{kl} the stress tensor, M_{kl} the couple stress tensor, f_k the body force per unit mass, l_k the body couple per unit mass and σ_k the spin inertia per unit mass (microrotational microacceleration), which is connected to microinertia and microgyration via the relation

$$\sigma_k = \overbrace{I_{kl} v_l} \quad (21)$$

In analogy to rigid body mechanics the spin inertia can be viewed as the time derivative of the angular momentum $I_{kl} v_l \approx I_{kl} \dot{\phi}_l$, where $\dot{\phi}_l$ is the rotational velocity defined above, i.e. $\dot{\sigma}_k = \dot{I}_{kl} \dot{\phi}_l + I_{kl} \ddot{\phi}_l$. Further, u is the internal energy per unit mass, s the entropy per unit mass, q_k the heat flux vector, Q the heat supply per unit mass and θ the absolute temperature. These balance laws are valid for all micropolar continua, whether solid, fluid or other. The balances of mass, linear momentum and entropy are standard. In the balances of angular momentum and energy extra terms appear in addition to those known from classical continuum theory (rational thermomechanics). The balance of microinertia is a special feature of micropolar theory and does not appear in the classical theory. Preliminary versions of micropolar balance laws are contained in a 1964 paper of Eringen and Suhubi [19] and in the early (1909) work of the Cosserat brothers [20]. The latter work, however, does not define a specific spin inertia and does not include the conservation law for the microinertia tensor. For the balance laws at discontinuous phase boundaries (jump conditions) the reader should consult Eringen's books [1,2]. In order to solve concrete engineering problems appropriate constitutive equations (corresponding to the material model chosen) should be inserted into the balance laws. As in classical continuum theory, this results in field equations which can finally be solved under initial and boundary conditions which have to be properly chosen to describe the process and geometry in question. The boundary conditions can be easily derived from the jump conditions, cf. [1,2]. The main task for materials scientists and engineers, however, is the proper selection and implementation of the constitutive equation (material model).

CONSTITUTIVE EQUATIONS OF MICROPOLAR SOLIDS

The nonlinear constitutive equations of micropolar thermoelastic solids can be formulated in terms of two potentials, the *free energy (Helmholtz potential)* Ψ for the static, i.e. reversible, parts of the constitutive equations,

$$\Psi = \Psi (C_{KL}, W_{KL}, \theta, I_{KL}), \quad (22)$$

and the *dissipation function* Φ for the dynamic, i.e. irreversible, parts of the constitutive equations,²

$$\Phi = \Phi (C_{KL}, W_{KL}, \theta, \theta_{,k}, I_{KL}). \quad (23)$$

For the stress tensor, the couple stress tensor, the heat flux vector and the entropy the following relations hold:

1. Stress tensor:

$$T_{kl} = \rho \frac{\partial \Psi}{\partial C_{KL}} \chi_{k,K} \chi_{lL} \quad (24)$$

2. Couple stress tensor:

$$M_{kl} = \rho \frac{\partial \Psi}{\partial W_{LK}} \chi_{k,K} \chi_{lL} \quad (25)$$

3. Heat flux vector:

$$q_k = \frac{\partial \Phi}{\partial \theta_{,k}} \quad (26)$$

4. Entropy:

$$s = -\frac{\partial \Psi}{\partial \theta} \quad (27)$$

The essential difference to classical continuum theory is the appearance of the couple stress tensor as a material response (dependent variable) and of Cosserat, wryness and microinertia tensors in the set of independent variables (instead of one of the classical strain measures).

When in a thermoelastic process deformations, rotations and temperature changes $(\Delta T) = \theta - \theta_0$ (from a reference temperature θ_0) are small, *linear approximation of the constitutive equations* is possible and often useful. Furthermore, linear deformation and strain measures are often sufficient in this case. Introducing the *linear deformation and rotation measures*

$$\varepsilon_{kl} \equiv u_{l,k} + \varepsilon_{ikm} \phi_m \quad (28)$$

and

$$\gamma_{kl} \equiv \phi_{k,l}, \quad (29)$$

where u_i is the displacement vector (familiar from classical linear continuum mechanics) and ϕ_m the rotation vector (given by $\phi_m = \phi_{nm}$), the *linear constitutive equations of micropolar thermoelastic solids* can be written as:

1. Stress tensor:

$$T_{kl} = -A_{kl}(\Delta T) + A_{klmn} \varepsilon_{mn} + C_{klmn} \gamma_{mn} \quad (30)$$

2. Couple stress tensor:

$$M_{kl} = -B_{lk}(\Delta T) + B_{lkmn} \gamma_{mn} + C_{mnlk} \varepsilon_{mn} \quad (31)$$

3. Heat flux vector:

$$q_k = K_{kl}(\Delta T)_{,l} \quad (32)$$

4. Entropy:

$$s = s_0 + C_0(\Delta T) + \frac{1}{\rho} A_{kl} \varepsilon_{kl} + \frac{1}{\rho} B_{lk} \gamma_{lk} \quad (33)$$

The material moduli C_0 (scalar), A_{kl} , B_{kl} , K_{kl} (second-order tensors) and A_{klmn} , B_{klmn} , C_{klmn} (fourth-order tensors) possess the following symmetries:

$$A_{klmn} = A_{mnkl}, \quad (34)$$

$$B_{klmn} = B_{mnkl}, \quad (35)$$

$$K_{kl} = K_{lk}. \quad (36)$$

Except for these symmetries no further symmetries can be a priori assumed, i.e. in the case of the most general anisotropic micropolar solid (micropolar triclinic crystal) the number of independent thermoelastic material moduli is 1, 9, 9, 45, 45 and 81 for C_0 , A_{kl} , B_{kl} , K_{kl} , A_{klmn} , B_{klmn} , and C_{klmn} , respectively, which makes a total of 196 scalar material moduli, cf. [1]. Note that the number of independent material moduli for triclinic crystals within the framework of classical linear thermoelasticity is 1, 6, 6 and 21 for the specific heat, the stress-temperature tensor (thermal expansion tensor), the heat conductivity tensor and the elasticity tensor, respectively, which makes a total of 34 scalar material moduli. In the formulation of these equations the reference state ("natural state") of the solid is assumed to be stress-free and couple-stress free. The dissipation potential is approximated by

$$\Phi = \frac{1}{2} K_{kl}(\Delta T)_{,k}(\Delta T)_{,l} \quad (37)$$

Similar to classical continuum mechanics, the linear approximation for the constitutive equations of stress, couple stress and entropy requires a quadratic approximation for the free energy (Helmholtz potential):

$$\begin{aligned} \Psi = & \Psi_0 - s_0(\Delta T) - \frac{C_0}{2}(\Delta T)^2 - \frac{1}{\rho} A_{kl} \varepsilon_{kl} - \frac{1}{\rho} B_{lk} \gamma_{lk} + \\ & + \frac{1}{2\rho} A_{klmn} \varepsilon_{kl} \varepsilon_{mn} + \frac{1}{2\rho} B_{lkmn} \gamma_{lk} \gamma_{mn} + \frac{1}{\rho} C_{klmn} \varepsilon_{kl} \gamma_{mn} \end{aligned} \quad (38)$$

Thus the linear constitutive equations can be derived from the free energy (Helmholtz potential) and the dissipation potential via the relations

1. Stress tensor:

$$T_{kl} = \rho \frac{\partial \Psi}{\partial \varepsilon_{kl}} \quad (39)$$

2. Couple stress tensor:

$$M_{kl} = \rho \frac{\partial \Psi}{\partial \gamma_{lk}} \quad (40)$$

3. Heat flux vector:

$$q_k = \frac{\partial \Phi}{\partial (\Delta T)_{,k}} \quad (41)$$

4. Entropy:

$$s = - \frac{\partial \Psi}{\partial (\Delta T)} \quad (42)$$

In the special case of *isotropic solids* the material moduli simplify as follows:

$$A_{kl} = \alpha_0 \delta_{kl} \quad (43)$$

$$B_{kl} = 0 \quad (44)$$

$$A_{klmn} = \lambda \delta_{kl} \delta_{mn} + (\mu + \kappa) \delta_{km} \delta_{ln} + \mu \delta_{kn} \delta_{lm} \quad (45)$$

$$B_{klmn} = \alpha \delta_{kl} \delta_{mn} + \beta \delta_{kn} \delta_{lm} + \gamma \delta_{km} \delta_{ln} \quad (46)$$

$$K_{kl} = K \delta_{kl} \quad (47)$$

$$C_{klmn} = 0 . \quad (48)$$

Thus the *linear constitutive equations of isotropic micropolar thermoelastic solids* are

1. Stress tensor:

$$T_{kl} = -\alpha_0 (\Delta T) \delta_{kl} + \lambda \varepsilon_{mm} \delta_{kl} + (\mu + \kappa) \varepsilon_{kl} + \mu \varepsilon_{lk} . \quad (49)$$

2. Couple stress tensor:

$$M_{kl} = \alpha \gamma_{mm} \delta_{kl} + \beta \gamma_{kl} + \gamma \gamma_{lk} . \quad (50)$$

3. Heat flux vector:

$$q_k = K (\Delta T)_{,k} . \quad (51)$$

4. Entropy:

$$s = s_0 + C_0 (\Delta T) + \left(\frac{\alpha_0}{\rho} \right) \varepsilon_{mm} . \quad (52)$$

The free energy (Helmholtz potential) is in this case:

$$\begin{aligned} \Psi = & \Psi_0 - s_0 (\Delta T) - \frac{C_0}{2} (\Delta T)^2 - \frac{\alpha_0}{\rho} (\Delta T) \varepsilon_{mm} \\ & + \frac{1}{2\rho} [\lambda \varepsilon_{kk} \varepsilon_{ll} + (\mu + \kappa) \varepsilon_{kl} \varepsilon_{kl} + \mu \varepsilon_{kl} \varepsilon_{lk}] \\ & + \frac{1}{2\rho} (\alpha \gamma_{kk} \gamma_{ll} + \beta \gamma_{kl} \gamma_{lk} + \gamma \gamma_{kl} \gamma_{kl}) \end{aligned} \quad (53)$$

It can be seen that the number of independent material moduli of linear thermoelastic solids in the frame-

work of micropolar theory is reduced from 196 (for the most general anisotropic case) to 9 (for isotropic solids), simply due to reasons of symmetry. Thus, linear isotropic micropolar thermoelastic solids are characterized by 6 elastic constants ($\lambda, \mu, \kappa, \alpha, \beta, \gamma$), one stress-temperature coefficient (thermal expansion coefficient) α_0 , one thermal conductivity K and one specific heat $C_0 \theta_0$. In the framework of classical continuum mechanics the number of independent material moduli of linear thermoelastic solids is five ($\alpha_0, \lambda, \mu, K, C_0$). Note that, similar to the classical case, the isotropic micropolar material moduli ($\alpha_0, \lambda, \mu, \kappa, \alpha, \beta, \gamma, K, C_0$) are functions of position for heterogeneous (multiphase) or non-uniform (gradient) materials. They are constants only for homogeneous (for the scale chosen) and uniform materials. Apart from the above material moduli, micropolar materials are characterized by the microinertia tensor I_{kl} . Micropolar materials for which this tensor has only one single component, i.e.

$$I_{kl} = I \delta_{kl} , \quad (54)$$

are called *microisotropic or spin-isotropic*. Microisotropy implies that the microinertia balance (conservation law) reduces to

$$\dot{I} = 0 . \quad (55)$$

Certain conditions (inequalities) must be met by the micropolar material moduli of isotropic solids in order to satisfy the postulate of thermodynamic stability [1]:

$$3\lambda + 2\mu + \kappa \geq 0 , \quad (56)$$

$$2\mu + \kappa \geq 0 , \quad (57)$$

$$\kappa \geq 0 , \quad (58)$$

$$3\alpha + \beta + \gamma \geq 0 , \quad (59)$$

$$\gamma + \beta \geq 0 , \quad (60)$$

$$\gamma - \beta \geq 0 , \quad (61)$$

$$C_0 \geq 0 , \quad (62)$$

$$K \geq 0 . \quad (63)$$

Moreover, the requirement of non-negative kinetic energy [1],

$$\frac{1}{2} \rho \dot{u}_k \dot{u}_k + \frac{1}{2} \rho I \dot{\phi}_k \dot{\phi}_k \geq 0 , \quad (64)$$

where u_k is the displacement vector and ϕ_k the microrotation vector $\phi_k = \phi n_k$ (cf. the preliminary section above), implies that

$$\rho \geq 0 , \quad (65) \quad I \geq 0 . \quad (66)$$

CONSTITUTIVE EQUATIONS
OF MICROPOLAR FLUIDS

In analogy to solids, the nonlinear constitutive equations of micropolar thermoviscous fluids can be formulated in terms of two potentials, the free energy Ψ (for the static, i.e. reversible, parts of the constitutive equations),

$$\Psi = \Psi(\theta, \rho^{-1}, I_{kl}), \quad (67)$$

and the dissipation function Φ (for the dynamic, i.e. irreversible, parts of the constitutive equations),

$$\Phi = \Phi(a_{kl}, b_{kl}, \theta, \theta_{,k}, \rho^{-1}, I_{kl}). \quad (68)$$

For the stress tensor, the couple stress tensor, the heat flux and the entropy the following constitutive equations are obtained.

1. Stress tensor:

$$T_{kl} = -\pi \delta_{kl} + \frac{\partial \Phi}{\partial a_{kl}}, \quad (69)$$

where π is the thermodynamic pressure given by

$$\pi = -\frac{\partial \Psi}{\partial \rho^{-1}}. \quad (70)$$

Note that for incompressible fluids (i.e. liquids) the pressure is an indeterminate hydrostatic pressure p of arbitrary magnitude, instead of the thermodynamic pressure π , which is given by an equation of state.

2. Couple stress tensor:

$$M_{kl} = \frac{\partial \Phi}{\partial b_{lk}}. \quad (71)$$

3. Heat flux vector:

$$q_k = \frac{\partial \Phi}{\partial \theta_{,k}}. \quad (72)$$

4. Entropy:

$$s = -\frac{\partial \Psi}{\partial \theta}. \quad (73)$$

The constitutive equations of heat flux and entropy are identical for solids and fluids. The essential difference to classical continuum theory is again the occurrence of the couple stress tensor, as well as the appearance of the microinertia tensor I_{kl} and the micropolar deformation-rate tensors a_{kl} and b_{kl} in the set of independent variables (instead of the symmetric part of the velocity gradient [18]).

The *linear constitutive equations of anisotropic micropolar fluids* are [2]

1. Stress tensor:

$$T_{kl} = -\pi \delta_{kl} + \alpha_{ijk} a_{ij}. \quad (74)$$

2. Couple stress tensor:

$$M_{kl} = \beta_{ijk} b_{ij} + d_{lk} \theta_{,j}. \quad (75)$$

3. Heat flux vector:

$$q_k = K_{kj} \theta_{,j} + d_{ijk} b_{ij}. \quad (76)$$

4. Entropy:

$$s = -\frac{\partial \Psi}{\partial \theta}. \quad (77)$$

The free energy Ψ is a function of θ , ρ^{-1} and the scalar invariants of the microinertia tensor I_{kl} , e.g. $\text{tr} I_{kl}$, $\text{tr}(I_{kl}^2)$, $\text{tr}(I_{kl}^3)$:

$$\Psi = \Psi[\theta, \rho^{-1}, \text{tr} I_{kl}, \text{tr}(I_{kl}^2), \text{tr}(I_{kl}^3)]. \quad (78)$$

Similarly, the dissipation function Φ is a function of a_{kl} , b_{kl} , $\theta_{,k}$, θ , ρ^{-1} and 11 scalar invariants (cf. [2], p. 108). Note that the thermodynamic pressure π and the entropy s are now also functions of the three invariants $\text{tr} I_{kl}$, $\text{tr}(I_{kl}^2)$, $\text{tr}(I_{kl}^3)$.

The tensorial material moduli of anisotropic micropolar fluids are

$$\begin{aligned} \alpha_{ijkl} = & \alpha_1 \delta_{ij} \delta_{kl} + \alpha_2 \delta_{il} \delta_{jk} + \alpha_3 \delta_{ik} \delta_{jl} + \\ & + \alpha_4 (I_{ik} \delta_{ij} + I_{ij} \delta_{kl}) + \alpha_5 (I_{ii} \delta_{jk} + I_{jk} \delta_{ii}) + \\ & + \alpha_6 (I_{jl} \delta_{ik} + I_{ik} \delta_{jl}) + \alpha_7 (I_{ji} \delta_{ik} - I_{ik} \delta_{ji}) \end{aligned} \quad (79)$$

$$\begin{aligned} \beta_{ijkl} = & \beta_1 \delta_{ij} \delta_{kl} + \beta_2 \delta_{il} \delta_{jk} + \beta_3 \delta_{ik} \delta_{jl} + \\ & + \beta_4 (I_{ik} \delta_{ij} + I_{ij} \delta_{kl}) + \beta_5 (I_{ii} \delta_{jk} + I_{jk} \delta_{ii}) + \\ & + \beta_6 (I_{jl} \delta_{ik} + I_{ik} \delta_{jl}) + \beta_7 (I_{ji} \delta_{ik} - I_{ik} \delta_{ji}) \end{aligned} \quad (80)$$

$$d_{ijk} = d_1 \varepsilon_{ijk} + d_2 \varepsilon_{ij} I_{lk}, \quad (81)$$

$$K_{ij} = K_1 \delta_{ij} + K_2 I_{ij}. \quad (82)$$

Since all material tensors depend on the microinertia tensor I_{kl} , they are subject to change with the motion of the fluid, i.e. with its microstructural evolution during flow (governed by the microinertia conservation law). For isotropic micropolar fluids the microinertia tensor is isotropic, i.e.

$$I_{kl} = I \delta_{kl}. \quad (83)$$

In this case the balance law for microinertia (conservation of microinertia) reduces to the equation

$$I = 0 \quad (84)$$

and the material moduli do not change during flow.

The *linear constitutive equations of isotropic micropolar fluids* are

1. Stress tensor:

$$T_{kl} = \lambda a_{mm} \delta_{kl} + (\mu + \kappa) a_{kl} + \mu a_{lk}. \quad (85)$$

2. Couple stress tensor:

$$M_{kl} = \beta_0 \epsilon_{klm} \theta_{,m} + \alpha b_{mn} \delta_{kl} + \beta b_{kl} + \gamma b_{lk}. \quad (86)$$

3. Heat flux vector:

$$q_k = K \theta_{,k} + \beta_0 \theta \epsilon_{klm} v_{m,l}. \quad (87)$$

In these equations λ , μ , κ are the viscosity coefficients for the stress tensor (translational viscosities), α , β , γ are the new viscosity coefficients responsible for gyrational dissipation (rotational viscosities) and K is the thermal conductivity. All these coefficients are functions of the specific volume ρ^{-1} and the temperature θ . The new terms involving β_0 correspond to a possible heat conduction due to microrotation and the possible generation of couple stresses due to temperature gradients. These two effects are not allowed in classical fluids. Similar to isotropic micropolar solids, thermodynamic stability requires certain conditions (inequalities) to be fulfilled for isotropic micropolar fluids [2], cf. Eqs. (56) through (63).

Evidently, micropolar fluid mechanics reduces to classical fluid mechanics when $I_{kl} = 0$, $\alpha = \beta = \gamma = \kappa = 0$ and the body couples vanish ($l_k = 0$), cf. [11]. In contrast to Newtonian fluids (Navier-Stokes fluids) linear micropolar fluids are characterized by 6 Reynolds numbers. Moreover, micropolar fluids possess two internal characteristic lengths, a static one (L_1) and a dynamic one (L_2):

$$L_1 = \sqrt{\frac{\gamma}{\kappa}}, \quad (88)$$

$$L_2 = \sqrt{I}. \quad (89)$$

APPLICATIONS

According to Eringen [1,2] the range of possible materials to be modelled by micropolar theory is very wide. It encompasses e.g., in Eringen's own words, "anisotropic fluids, liquid crystals with rigid molecules, rigid suspensions, magnetic fluids, clouds with dust, muddy fluids, biological fluids, animal blood with rigid cells, chopped fiber composites, bones, concrete with sand". Although disputable from a logical point of view,³ this enumeration shows how wide is the range of intended applications of micropolar theory.

Of course, the experimental effort necessary to obtain the required material parameters is often an unsurpassable barrier. As Eringen [1] states, "experiments with micropolar constants require much precision and elaborate instrumentation, since we are faced with the measurement of microscopic-level quantities in high-frequency, short-wavelength regions." Therefore, experimental work using micropolar theory as a basis for the

design of experimental setups (in order to measure at least some of the many material parameters) and the interpretation of measured data is still very rare. Nevertheless, a few promising results are available in the literature, and we mention some of these.

In the case of *micropolar solids* it is useful to distinguish two classes of materials, those with periodic microstructure and those *with random microstructure*. The application of micropolar theory to solids with *periodic microstructure* (or "structure" in general) is extensive and rather successful. It ranges from natural materials with lattice structure (crystals) to man-made composite materials and engineering structures such as infinite-fiber composites, sandwich structures, grid structures, trusses and honeycombs. Examples of the successful application of micropolar theory to these materials are the calculation of micropolar elastic moduli of KNO_3 and other crystals (especially ferroelectric ones) exhibiting polar phenomena by Askar [21], Fischer-Hjalmar [22,23] and Pouget et al. [24] and the numerical analysis of steel-concrete grid structures in civil engineering, performed e.g. by Bažant and Christensen [25].

The application of micropolar theory to solids *with random microstructure*, natural or man-made (chopped-fiber composites, platelet composites, particulate composites, porous materials, foams, bone), although useful in principle, is very complicated in practice. Rather non-trivial experimental measurements have to be performed with high precision in order to obtain at least some of the many micropolar material moduli. The difficulty of performing experimental measurements is caused by the fact that micropolar effects are observable only when the internal characteristic length (for isotropic materials e.g. the ratio $\sqrt{\gamma/G}$, where γ is one of the material parameters occurring in Eq. (50), with units [N], and $G = \mu + \kappa/2$ is the micropolar shear modulus, with units [$\text{Pa} = \text{N/m}^2$]) is in the order of the external characteristic length (e.g. the length scale of the material heterogeneities, such as crack tip radius, grain size, pore size in static experiments or a wavelength in dynamic experiments). Moreover, the symmetry of the material has to be determined by an independent method before experimental data can be interpreted in terms of micropolar theory.

Gauthier [26] performed dynamical (wave propagation) experiments on a particulate composite consisting of aluminum particles dispersed in an epoxy resin matrix. For this composite he succeeded in determining the micropolar elastic moduli $\lambda = 7.59$ GPa, $\mu = 1.89$ GPa, $\kappa = 0.0149$ GPa, $\gamma = 2.63$ kN, $I = 0.196$ mm². The determined characteristic length of 0.76 mm turned out to be very close to the radius of the aluminum particles (approx. 0.7 mm).

Yang and Lakes [27] determined the elastic properties of human compact bone and interpreted their results

in terms of micropolar theory (under the assumption $\beta/\gamma=1$). Their value for the micropolar tensile modulus (Young's modulus),

$$E \equiv \frac{(2\mu + \kappa)(3\lambda + 2\mu + \kappa)}{2\lambda + 2\mu + \kappa}, \quad (90)$$

is 14.4 (± 3.3) GPa and the characteristic length is determined to be 0.49 (± 0.18) mm, rather close to the typical diameter of osteons (approx. 0.25 mm), indicating the significance of osteons as structural elements of bone. Yang and Lakes' investigation showed, inter alia, that the prediction of Eringen's micropolar theory (14.4 GPa) corresponds better to the experimentally determined stiffness than the prediction of Toupin's [28] and Mindlin's and Thiersten's [29] couple stress theory (17.0 GPa), cf. also [30]. Other applications of micropolar theory to bone can be found e.g. in [31,32]. For further applications of micropolar theory to solids, including the modelling of grain size effects and wave propagation, the reader may consult [1,33-38] and the references cited therein.

The theory of *micropolar fluids* has found several applications in the description of physical phenomena which cannot be exhaustively explained by classical fluid mechanics. Among these are liquid crystals and the occurrence of *turbulence*. While the first is beyond the scope of this paper, we mention a few facts about the latter. In classical fluid mechanics, the generation of vortices is attributed to viscous friction. When there is no macroscopic motion (i.e. the velocity field is zero) the fluid has no vortices. The process of vortex creation remains unexplained. Contrary to this, in micropolar fluid mechanics vortices can exist even in the absence of a macroscopic velocity field, and it is hypothesized by Eringen [2], that the hidden microrotation field is the source for the creation of vortices. For the complete micropolar description of turbulence and many other applications to problems of micropolar fluid mechanics the reader should consult [2,11,39-41] and many papers in the International Journal of Engineering Science, e.g. [42-44]. In these references the reader will find applications such as flow in a non-coaxial plate-plate rheometer, lubrication problem (generalized Reynolds equation), Stokes flow about a sphere, stagnation flow, Taylor-Bénard instability, boundary layer flow over a plate, mixed convection in vertical flow and flow of micropolar fluids in porous media (generalized Darcy equation). Also body fluids and biological flow problems have been modelled by micropolar theory [45,46]. Micropolar fluid lubrication with reference to human joints is discussed in [47,48]. Trivially, the theory of micropolar fluids contains the couple stress theory for fluids as well as the classical theory of Newtonian fluids (Navier-Stokes fluids) as special cases.

Although less elaborated for that case, micropolar fluid theory can be applied to anisotropic fluids as well, as mentioned above. Clearly, it must remain a desideratum of further investigations to perform a precise comparison between Eringen's theory of micropolar anisotropic fluids and other theories of anisotropic liquids, e.g. Ericksen's theory [17]. First steps have been done in [49]. It is evident, however, that Eringen's theory is based on a broader and more general fundament than any other present theory in this field. Therefore it is highly probable that micropolar theory, when formulated in sufficient generality, can account for the essential parts of Ericksen's theory of anisotropic fluids as well as for other more recent theories, in particular the current theory of fiber suspensions [50]. Of primary importance would be a rational answer to the question, in what way the microstructure tensors (orientational tensors) occurring in the latter theories are related to the tensors occurring in the micropolar theory.

In this connection it should be noted that, according to Eringen, the case of *suspensions* requires special attention and cannot be treated in a standard way. For suspensions, Eringen proposes a modified balance law for microinertia [2,8], viz.

$$I_{kl} + I_{km}v_{ml} + I_{lm}v_{mk} - F_{ijkl}a_{ij} = 0, \quad (91)$$

where v_{kl} is the microgyration tensor and F_{ijkl} is a fourth-order tensor, related to the evolution of fluid (suspension) microstructure during flow. In the case of anisometric particles (fibers or platelets) the degree of orientation can change during flow, leading to changes in the material moduli, including viscosities, and their symmetry. Eringen's argument for introducing this fundamental modification is that in the case of suspensions motion causes non-smooth macroscopic fields, i.e. the velocity difference between microstructure (suspended particles) and macrostructure (macroscopic flow field), gives rise to additional contributions to the microinertia balance law.

For *rigid fiber suspensions* (fibers modelled as bars/rods or prolate spheroids) the microinertia tensor can be expressed as

$$I_{kl} = I(\delta_{kl} - n_k n_l), \quad (92)$$

where n_k is a unit vector denoting fiber orientation. In this case the microinertia balance law takes the form

$$\dot{n}_k + n_i v_{ik} + \alpha_{fiber}(a_{ij}n_i n_j n_k - a_{(kl)}n_l) + \beta_{fiber}a_{[kl]}n_l = 0, \quad (93)$$

where α_{fiber} and β_{fiber} are material constants and $a_{(kl)}$ and $a_{[kl]}$ are the symmetric and antisymmetric parts of a_{ij} , respectively.

For *rigid sphere suspensions* the microinertia tensor is isotropic, i.e.

$$I_{kl} = I\delta_{kl}, \quad (94)$$

and the microinertia balance law takes the form

$$I = 0 \quad (95)$$

In [2] Eringen treats channel flow and Couette flow of rigid fiber suspensions. For two-dimensional channel pressure-driven flow with a velocity gradient in the y -direction a closed-form solution is presented for the evolution of the orientation angle φ in dependence of the fiber aspect ratio R [2,9]. When the fibers start with a preferential orientation (parallel to the flow direction x) this solution is

$$\tan \varphi = \sqrt{\frac{1 - \alpha_{fiber}}{1 + \alpha_{fiber}}} \tan \left(\frac{\sqrt{1 - \alpha_{fiber}}}{2} \cdot \frac{xy}{1 - y^2} \right), \quad (96)$$

where the material coefficient α_{fiber} is connected to the aspect ratio R (length-diameter ratio) via the relation

$$\alpha_{fiber} = \frac{R^2 - 1}{R^2 + 1} < 1 \quad (97)$$

From this solution it can be concluded that fibers, starting in horizontal position, become vertical at a critical distance x_c , given by

$$x_c = \frac{\pi}{\sqrt{1 - \alpha_{fiber}^2}} \cdot \frac{1 - y^2}{y} \quad (98)$$

In terms of aspect ratio the solutions are

$$\tan \varphi = \frac{1}{R} \tan \left(\frac{R}{R^2 + 1} \cdot \frac{xy}{1 - y^2} \right) \quad (99)$$

and the critical distance x_c is given by

$$x_c = \frac{\pi}{2} \cdot \frac{R^2 + 1}{R} \cdot \frac{1 - y^2}{y} \quad (100)$$

At distances $x > x_c$ the fibers proceed to rotate. A consideration of critical distances in dependence of y shows that along the centerline of the channel the horizontal positions of the fibers are not changed, while near the walls the fibers rotate to vertical positions immediately. A similar result is reported for Couette flow of fiber suspensions: For a fixed tangential angle (i.e. following a radial line) the fiber orientation angle φ increases until it is perpendicular to the radial line (at a critical distance r_c). For $r > r_c$ the fiber orientation is the reverse of that for $r < r_c$. The periodic rotating motion of fibers in shear flow corresponds to the so-called Jeffery orbits which have been predicted theoretically [51] and found experimentally [50].

Note that, apart from *flow-induced orientation effects* (which are naturally absent in sphere suspensions), flow can induce non-uniformity in the suspen-

sion due to *flow-induced particle migration*, for suspensions with spherical or anisometric particles [52]. This phenomenon is not accounted for in micropolar theory. The reason is that micropolar theory is designed to describe the behavior of materials with microstructure (including two-phase mixtures, e.g. suspensions with anisometric particles) without being a mixture theory *sensu strictu*, i.e. without even invoking volume fractions. Nevertheless, micropolar theory seems to be the most appropriate candidate for a reconciliation of the competing theories for the description of anisotropic fluids with microstructure, including suspensions with anisometric particles, into a unified framework.

CONCLUSION

One of the most famous non-classical continuum theories for materials with microstructure, Eringen's theory of micropolar continua, has been summarized in a concise form and reviewed in its state-of-the-art. Based on Eringen's "directors", kinematic measures have been introduced for solids and fluids (micropolar strain and strain-rate tensors C_{KL} , W_{KL} , a_{kl} , b_{kl}). The balance equations have been presented, including the non-classical balance law for the microinertia tensor I_{kl} . Non-linear and linear constitutive equations (material models) have been given for anisotropic and isotropic solids, as well as anisotropic and isotropic fluids, respectively. The physical significance of the material parameters has been explained and the reduction of their number due to symmetry discussed. It has been shown that, although (due to the difficulty of the pertaining experiments) applications of micropolar theory are still relatively rare (especially for random microstructures), micropolar theory should be a useful framework for modeling solid composites (fiber, platelet, or particulate), porous media (including solids with fissures or microcracks) and suspensions (containing isometric or anisometric particles).

In particular, it can be expected that the mechanical properties of the majority of ceramic systems (elasticity of ceramic composites and porous ceramics, viscosity of ceramic suspensions) with anisometric microstructural features (particles or pores) can be more adequately described by micropolar theory than by any other theory currently available. Although the experimental determination of the additional material parameters may be a non-trivial task, this difficulty will probably be outweighed by the advantages of a possibly higher precision of the predictions and the embedding of anisotropic systems (including the case of anisotropic fluids, by ceramicists traditionally considered to be a rather exotic topic) in a natural, aesthetically appealing and sufficiently general theoretical framework.

Footnotes:

¹ Eringen [1] calls "directors" also "two-point tensors". In contrast to usual tensors, but similar to the deformation gradient, directors correspond to a mapping between two different frames of reference. In spite of their name, directors should not be confused with orientational vectors, used e.g. by Ericksen [17] in his theory of anisotropic fluids.

² In order to retain as far as possible the structure of classical thermodynamics, the temperature change rate $\dot{\theta}$ is omitted a priori from the set of independent variables of the dissipation function here, in contrast to [1,2].

³ This enumeration combines two different logical categories, viz. material models (such as "anisotropic fluids") and real materials (such as "concrete with sand"). Further, some categories, e.g. "biological fluids", are rather undefined and some of these should rather be modelled, as the case may be, as microstretch or micromorphic materials. Last but not least, as for all material models, also the micropolar material model may be adequate for a certain real material in one process but not in another.

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MIKROPOLÁRNÍ MATERIÁLY

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V této práci je shrnuta Eringenova teorie mikropolárních kontinuí pro materiálové vědce a inženýry. Jsou zavedeny kinematické míry jak pro pevné látky tak pro tekutiny (mikropolární tenzory deformace resp. rychlosti deformace) a uvedeny bilanční rovnice, včetně bilanční rovnice pro tenzor mikrosetrvačnosti, který se v klasických teoriích neobjevuje. Nelineární a lineární konstitutivní rovnice (materiálové modely) jsou prezentovány pro anizotropní a izotropní pevné látky a tekutiny. Je vysvětlen fyzikální význam materiálových parametrů a ukázáno, jak se jejich počet redukuje důsledkem symetrie. Na základě příkladů z literatury je prezentováno, že mikropolární teorie může být užitečným rámcem pro modelování pevných kompozitů, porézních medií a suspenzí.