AN OUTPUT FEEDBACK ALGORITHM FOR TRAJECTORY TRACKING IN CONTROL AFFINE NONLINEAR SYSTEMS

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***Abstract*— In this work we present an output feedback algorithm that solves the trajectory tracking problem in control affine nonlinear systems. This algorithm, is an improvement, for this class of systems, of that of (Mancilla Aguilar *et al.* 2000a), since it reduces the chattering effect on the control while keeping the original performance. In addition, and via a high gain observer, it deals with discrete output measurements instead of the states, as the original algorithm does.**

***Keywords*— Sampled-data; Chattering; Observer; Output feedback; Trajectory Tracking.**

I. INTRODUCTION

Nowadays, the use of digital computers for the control of continuous-time systems is commonplace. For this reason it is important to study the digital implementation of continuous-time control laws. In this case it is assumed that the states (or outputs) of the system to be controlled are available only at certain times (sampled-data). Since the control is based on these sampled-data, during the intersample periods the control actions applied to the system will be open-loop ones, even if the original continuous-time control law is a feedback one. It is then natural to study the digital implementation of the diverse continuous-time control laws for nonlinear systems, in particular stabilizing or, more generally, trajectory tracking control laws.

Although good results are obtained via the digital implementation via Sample and Zero Order Hold (SZH) of stabilizing laws (see (Mancilla Aguilar *et al.*, 2000b) and the references therein for details), this is not the case for trajectory tracking unless strong assumptions about the tracking control law are made. An example presented in (Mancilla Aguilar *et al.*, 2000a) shows that there is no reason to expect a "nice" behavior of the implementation via SZH of a trajectory tracking control feedback law.

The algorithm proposed in (Mancilla Aguilar *et al.*, 2000a), from now on *Algorithm 0*, for a rather general class of systems, solves this problem by making the system to follow the trajectories of a model instead of discretizing the continuous-time tracking law. The control law so obtained assures semiglobal practical stability of the tracking error, with final error arbitrarily small for a small enough sampling period. The controller is robust with respect to external disturbances and actuator and data measurements errors, when all of them are small enough.

One of the main drawbacks of Algorithm 0, when applied to control affine systems, appears in the implementation. In fact, as the algorithm is based in a maximization process, when the control space is a polytope, the extremal control values appear at the vertexes. This fact usually gives origin to a *chattering effect* : the values of the control switch undesirably fast. Another drawback of the algorithm is that it makes use of the states of the system, and in general only discrete-time samples of the output are available.

In this work we develop, for control affine systems, a trajectory tracking algorithm that reduces the chattering effect appearing in Algorithm 0, while it keeps the original tracking error performance. It works with the output sampled-data due to the addition of a high gain observer developed by García *et al.*, (2000), that performs the state estimation.

The paper is organized as follows. In section II we introduce some basic definitions; in section III Algorithm 0 (adapted to control affine systems) is presented and an example where the chattering effect can be observed is shown. In section IV we present a modified algorithm that obtains a high reduction of the chattering effect, while a small tracking error is kept. In section V we review a high gain observer introduced in (García *et al.* 2000) and present the output feedback algorithm that copes with the two drawbacks mentioned above. In the same section, we apply this algorithm to the example of section III. Finally, the conclusions are presented in section VI.

II. NOTATION AND BASIC DEFINITIONS

Let us start with some basic definitions: $\mathbb{R}_{\geq 0}$ denotes the real interval $[0, +\infty)$. The continuous function $\gamma : [0, r) \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{K}$ if it is strictly increasing and satisfies $\gamma(0) = 0$; γ is *of class* $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$, is defined in $[0, \infty)$ and $\lim_{s\to\infty} \gamma(s) = \infty$. A continuous function $\beta : [0, r) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *of class* $\mathcal{KL}$ if for each fixed t the mapping $\beta(s, t)$ is of class $\mathcal{K}$ and for each fixed s, $\beta(s, t)$ is decreasing to zero on t as $t \to \infty$ (Khalil, 1995).

We use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the Euclidean inner product and the Euclidean norm in $\mathbb{R}^m$ whichever be m, respectively. A function $\omega : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^p$ is a *feedback control law* if for each $x \in \mathbb{R}^n$, $\omega(\cdot, x)$ is measurable and for each $t \geq 0$, $\omega(t, \cdot)$ is continuous.

Consider a nonlinear control system described by:

$$\begin{cases} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{cases} \tag{1}$$

with $x \in \mathbb{R}^n$ the state, $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$ the control, $y \in \mathbb{R}^p$ the output, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g = (g_1, \cdots, g_m)$ with $g_i : \mathbb{R}^n \to \mathbb{R}^n$ are locally Lipschitz and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the output function. Since we suppose that only the outputs are available, we consider that there exist an estimation $\hat{x}(t)$ for the state vector $x(t)$, (as might be given by an observer):

$$\hat{x}(t) = x(t) + d(t) \tag{2}$$

where the estimation error $d(t)$ satisfies $\|d(t)\| \leq \hat{d}, \quad \forall t \geq 0$.

We say that $x^* : [\tau_0, +\infty) \to \mathbb{R}^n$ is an *admissible trajectory (a reference)* for (1) if there exists a bounded measurable function (*its generator*) $u^* : [\tau_0, +\infty) \to \mathbb{R}^m$ such that $x^*(t)$ is a solution of $\dot{x}(t) = f(x(t)) + g(x(t))u^*(t)$ in a compact set of $\mathbb{R}^n$.

Definition II.1 *Let $x^* : [\tau_0, +\infty) \to \mathbb{R}^n$ be a given reference. We say that a feedback control law $\omega(t, x)$ solves the closed-loop uniform asymptotic problem (is a CLU for x^*) if the equation of the tracking error $e(t) = x(t) - x^*(t)$*

$$\dot{e}(t) = f(x^*(t) + e(t)) - f(x^*(t)) + \\ g(x^*(t) + e(t))\omega(t, x^*(t) + e(t)) - g(x^*(t))u^*(t)$$

has unique solution for each initial condition and has the origin as an uniformly globally asymptotically stable equilibrium, i. e.*, there exists $\beta \in \mathcal{KL}$ such that $\forall\, t \geq \tau_0,\ \|e(t)\| \leq \beta(\|e(\tau_0)\|, t - \tau_0)$.*

We will make in the sequel the following assumption, which is instrumental in order to assure the convergence of Algorithm 0 (see Mancilla Aguilar *et al.*, 2000a for details):

H1 $\omega(t, x)$ is such that there exists a non-decreasing function $\Phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\|\omega(t, x)\| \leq \Phi(\|x - x^*(t)\|)$;

and we will also adopt the following notation:

Given a sampling period δ, consider the sampling instants $\tau_k = k\delta, k = 0, 1, \cdots$, and for a positive real number T, let r be the first natural number such that $T(\delta) := r\delta \geq T$; then we denote $T_N(\delta) = NT(\delta)$ if $N \in \mathbf{N}$.

III. A TRAJECTORY TRACKING ALGORITHM

In this section we present Algorithm 0 adapted to control affine nonlinear systems described by (1).

Let fix a compact set $U \subset \mathbb{R}^m$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to U$, be a function that verifies:

$$< z - x, g(x)\varphi(z, x) >= \max_{u\in U} < z - x, g(x)u > . \tag{3}$$

Consider the control law defined for each pair $(z(\tau_k), \hat{x}(\tau_k))$ by:

$$u(t) = u(\tau_k) = \varphi(z(\tau_k), \hat{x}(\tau_k)) \quad \tau_k \leq t < \tau_{k+1}. \tag{4}$$

Then Algorithm 0 can be described in this case as follows:

1. In the interval $[T_N(\delta), T_{N+1}(\delta))$ we solve the initial value problem:

$$\begin{cases} \dot{z}(t) = f(z(t)) + g(z(t))\omega(t, z(t)) \\ z(T_N(\delta)) = \hat{x}(T_N(\delta)); \end{cases} \tag{5}$$

2. For $T_N(\delta) \leq \tau_k \leq t < \tau_{k+1} \leq T_{N+1}(\delta)$, the control $u(t)$ that we apply to the plant (1) is given by (4) with $z(t)$ as in (5), and $\hat{x}(\tau_k)$ and $z(\tau_k)$ are obtained from (2) and (5) respectively.

The next result, that appears in (Mancilla Aguilar *et al.*, 2000a), shows that the final tracking error can be made arbitrarily small for a suitable choice of the sampling period δ.

Theorem III.1 *Let x^* a reference for (1), $\omega(t, x)$ a CLU for it that verifies* **H1** *and positive real numbers R_0 and $\varepsilon_0 > 0$. Then there exist a compact set $U \subset \mathbb{R}^m$ and positive numbers T, δ_0, $\hat{d}$ and T' such that if $0 < \delta \leq \delta_0$ and $x(\cdot)$ is a trajectory of (1) corresponding to the control $u(\cdot)$ given by Algorithm 0, with $\|x(0) - x^*(0)\| \leq R_0$, we have:*

1. *there exist a $\mathcal{K}$-class function Δ, depending only on β such that $\|x(t) - x^*(t)\| \leq \Delta(R_0 + \varepsilon_0) + \varepsilon_0 \quad \forall t \geq 0$*
2. *$\|x(t) - x^*(t)\| \leq \varepsilon_0$ if $t \geq T'$.*

Although the maximization (3) over a compact set technically sounds, in practice the convex hull of a finite number of points is adopted as the control space, since the evaluation of $u(\tau_k)$ is easier and its value is unique. We consider then, $U = \prod_{j=1}^{m} [-a_j, a_j]$. In this case Eqn.(3) becomes: $< z - x, g(x)\varphi(z,x) > = \max_{u\in U} \sum_{j=1}^{m} < z - x, g_j(x)u_j >$.

Then, if $b_j(\tau_k) =< z(\tau_k) - \hat{x}(\tau_k),\ \ g_j(\hat{x}(\tau_k)) >$, (6)

the control u given by (4) has now components

$$u_j(t) = \begin{cases} -a_j & \text{if } b_j(\tau_k) < 0 \\ a_j & \text{if } b_j(\tau_k) \geq 0 \end{cases} \tag{7}$$

for $j = 1, \cdots, m$, and $t \in [\tau_k, \tau_{k+1})$.

This control strategy strongly resembles that of the variable structure controllers. In consequence, it is not surprising that a chattering effect may appear, as it is shown in the following simulation.

Consider the control affine system whose state $x = (x_1, x_2)$ is supposed to be available:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) + u_1(t)(1 - x_1^2(t)) \\ \dot{x}_2(t) &= -x_1(t) + u_2(t)(1 - x_2^2(t)) \end{cases} \tag{8}$$

The reference is: $\{x^*(t) = (x_1^*(t), x_2^*(t)), t \in [0, +\infty) : \|x^*(t)\| = r^*\}$ with $r^* \in (0,1)$. The feedback tracking law is: $\omega(t,x) = (\omega_1(t,x), \omega_2(t,x))$, with $\omega_1(t,x(t)) = \dfrac{-Kx_1(t)e_r(t)^2}{(1-x_1^2(t))\|x(t)\|^2}$ and $\omega_2(t,x(t)) = \dfrac{-Kx_2(t)e_r(t)^2}{(1-x_2^2(t))\|x(t)\|^2}$, where $e_r(t) = \|x(t)\| - r^*$ is the radial tracking error. This law verifies the assumption **H1** with $\Phi(\mu) = \hat{k}\mu^2$ and $\hat{k}$ a certain constant.

Then, $e_r(t)$ verifies the equation: $\dot{e}_r(t) = -Ke_r(t)^2$ where K is taken such that $sign(K) = sign(e_r(0))$, and then the closed loop system will track the reference with an asymptotic decaying error norm given by: $|e_r(t)| = \beta(|e_r(0)|, t)$, where $\beta(r,s) = r/(Krs+1)$.

In the simulations of the application of Algorithm 0 to this example, we adopted the values: $T = 2$; $\delta = 0.04$; $u = (u_1, u_2) \in [-1,1] \times [-1,1]$ and $K = -7$, and the initial conditions $(x_1(0), x_2(0) = (0.2, 0)$. Figures 1 and 2 show the results of the simulations. Figure 1 shows $|e_r(t)|$ (the blurred curve) and $|e_{rz}(t)| = |\|z(t)\| - r^*|$, the modulus of the radial tracking error of the model of the system (the smoother curve) (see Mancilla Aguilar *et al.*, (2000a) for precisions about the model). Figure 2 shows a detail of the tracking controls $u = (u_1, u_2)$ as given by (7). It can be observed that the controls switch very fast between the values 1 and -1 for any given time interval (the chattering effect).

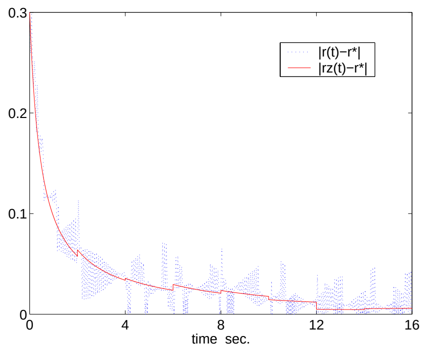

Figure 1: Algorithm 0: tracking errors.

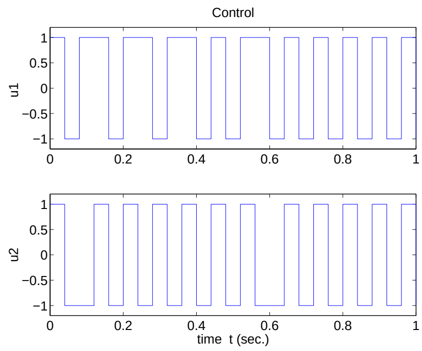

Figure 2: Algorithm 0: controls

IV. THE MODIFIED ALGORITHM

In what follows we introduce a modified algorithm that enables us to overcome the problem of chattering of Algorithm 0. The modifications are made in two steps: in the first one (Step A) a *boundary layer* is introduced in order to obtain a continuous approximation of the control (7). In the second step (Step B) a prediction strategy is added in order to improve the final tracking error performance of Step A.

• **Step A**

As can be easily seen from Eqn.(7), the change of each component u_j of the control is consequence of the change of the sign of b_j. In order to avoid the fast switching by smoothing out the control discontinuity, we introduce the following modification: given a positive value ϵ, if the value of b_j lies in the interval $[-\epsilon, \epsilon]$, u_j will take its values from a linear interpolation between $-a_j$ and a_j, and if $|b_j| > \epsilon$ it will take the values given by (7).

Then, for a given ϵ and $\tau_k \leq t < \tau_{k+1}$, the control law $u(t)$ will have now components defined by:

$$u_j(t) = \begin{cases} -a_j & \text{if } b_j(\tau_k) < -\epsilon \\ \frac{a_j}{\epsilon} \cdot b_j(\tau_k) & \text{if } |b_j(\tau_k)| \leq \epsilon \\ a_j & \text{if } b_j(\tau_k) > \epsilon \end{cases} \tag{9}$$

$j = 1, \cdots, m$, instead of those given by Eqn. (7).

Since Step A consists basically of a smoothing out of the original control discontinuity, it is to be expected that some degradation in the tracking performance with respect to that of Algorithm 0 appears. As a consequence, we introduce in addition a prediction strategy in order to improve the performance.

• **Step B**

The (one step ahead) strategy consists in achieving the value of b_j in (6) using *advanced* information give by Eqn. (5) (*prediction*). In this scheme, we replace $z(\tau_k)$ by $z(\tau_{k+1})$ in (6). The value of $z(\tau_{k+1})$ can be obtained from Eqn.(5) as

$$z(\tau_{k+1}) \cong z(\tau_k) + \delta \dot{z}(\tau_k) \cong z(\tau_k) + \delta[f(z(\tau_k)) + g(z(\tau_k))\omega(\tau_k, z(\tau_k))]. \quad (10)$$

In this way, the value of $b_j(\tau_k)$ is obtained as

$$b_j(\tau_k) =< z(\tau_{k+1}) - \hat{x}(\tau_k), \ \ g_j(\hat{x}(\tau_k)) > . \quad (11)$$

Then, for a boundary layer thickness ϵ, the modified algorithm (Algorithm 1) can be described as follows.

1. In the interval $[T_N(\delta), T_{N+1}(\delta))$ we solve the initial value problem (5).
2. For $T_N(\delta) \leq \tau_k \leq t < \tau_{k+1} \leq T_{N+1}(\delta)$, the control $u(t)$ that we apply to the plant is given by (9), b_j given by (11), $z(\tau_{k+1})$ as in (10) and where $\hat{x}(\tau_k)$ and $z(\tau_k)$ are obtained from (2) and (5) respectively.

Next, we state a lemma, which is analogous to Lemma 2.1 in (Mancilla Aguilar *et al.*, 2000a), and that can be proved similarly. This lemma shows the behaviour of control u as given by (9)-(11), for the tracking of a reference in a bounded interval of length T.

With this aim, for $K \subset \mathbb{R}^n$ a compact set and $\alpha = \max\{a_i, i = 1, \ldots, m\}$, let l_f and l_g the Lipschitz constants with respect to K for f and g respectively, and denote $m = \max_{u \in U, x \in K} \|f(x) + g(x)u\|$, $m_g = \max_{x \in K} \|g(x)\|$ and $\tilde{l} = (l_f + a l_g)$.

For any $\hat{d}, T, \epsilon$ and δ positive numbers, let us define $\Gamma_{K,\alpha}(T, \epsilon, \delta, \hat{d}) = \left[e^{2\tilde{l}T}\hat{d}^2 + \frac{(e^{2\tilde{l}T}-1)C}{2\tilde{l}} \right]^{1/2}$ with $C = 4\delta\alpha(2mm_g + ml_g) + 4\alpha\epsilon + 4\alpha\hat{d}(m_g + (2 + \hat{d} + \delta m)l_g)$ and $[K(z(.))]_1 := \{x \in \mathbb{R} : \exists t \in [0, T], \|x - z(t)\| \leq 1\} \subset \mathbb{R}^n$. The following holds:

Lemma IV.1 *Let $K \subset \mathbb{R}^n$ and α as above and pick $T, \delta, \hat{d}$ and ϵ such that $\Gamma_{K,\alpha}(T, \epsilon, \delta, \hat{d}) < 1$. Let $x^* : [\tau_{i^*}, \tau_{i^*} + T] \to \mathbb{R}^n$, with $\tau_{i^*} = i^*\delta$, be a trajectory of (1) such that $[K(x^*(\cdot))]_1 \subset K$ and whose generator $u^*(\cdot)$ verifies $u^*(t) \in U \ \ \forall t \in [\tau_{i^*}, \tau_{i^*} + T]$. Then if $x(\cdot)$ is a trajectory of (1) controlled by u as given by (9)-(11), such that $\|x(\tau_{i^*}) - x^*(\tau_{i^*})\| \leq \hat{d}$, we have:*

$$\|x(t) - x^*(t)\| \leq \Gamma_{K,\alpha}(T, \delta, \epsilon, \hat{d}), \quad (12)$$

$\forall t \in [\tau_{i^*}, \tau_{i^*} + T]$.

Remark IV.1 *Note that we can make the tracking error arbitrarily small by choosing a suitable δ, if $\hat{d}$ and ϵ are small enough.*

Remark IV.2 *If we choose the "boundary layer thickness" ϵ too small the chattering effect appears, while if it is too large the final tracking error will also be large. The selection of the optimal value of ϵ follows from an analysis of the magnitude of the control fields $\|g_j(x)\|$ as x evolves in the compact K.*

Remark IV.3 *Lemma IV.1 gives a bound of the difference $\|x(t) - x^*(t)\|$ at each time interval of length T. A bound for the tracking error in the whole tracking time-scale is presented in the next theorem. This result is analogous to Theorem III.1 and can be proved in the same way.*

Theorem IV.1 *Let x^*, $\omega(t, x)$, R_0 and ε_0 as in Theorem III.1. Then there exist positive numbers α, T, δ_0, $\hat{d}$, ϵ and T' such that if $0 < \delta \leq \delta_0$ and $x(\cdot)$ is a trajectory of (1) corresponding to the control $u(\cdot)$ given by Algorithm 1 with $\|x(0) - x^*(0)\| \leq R_0$, we have:*

1. *there exist a $\mathcal{K}$-class function Δ, depending only on β such that $\|x(t) - x^*(t)\| \leq \Delta(R_0 + \varepsilon_0) + \varepsilon_0 \ \ \forall t \geq 0$*
2. *$\|x(t) - x^*(t)\| \leq \varepsilon_0$ if $t \geq T'$.*

Remark IV.4 *If in addition ϵ is not too small, the resulting control $u(\cdot)$ presents no chattering effect.*

V. THE OUTPUT FEEDBACK ALGORITHM

As we stated in the Introduction, generally the values of the states are not available and an estimation of them is needed in order to obtain b_j as given by (11). With this aim, in this section we present a high gain observer introduced in (García *et al.*, 2000) for a nonlinear continuous time system that is well fitted for this purpose, since it is designed in order to estimate the states when the output measurement is a discrete time process.

According to the assumptions made in (García *et al.*, 2000) we suppose that system (1) verifies the following hypotheses.

- **H2**: There exist p integer numbers $\eta_1, \eta_2, ..., \eta_p$ that verify: $\Sigma_{i=1}^p \eta_i = n$, and a globally Lipschitz diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$, with ϕ^{-1} also globally Lipschitz such that by performing the nonlinear change of coordinates $w = \phi(x)$, the system (1) can be written as:

$$\begin{cases} \dot{w} = Aw(t) + \chi(w, u) \\ y(t) = Cw(t) \end{cases} \quad (13)$$

  where $A = diag\{A_1, ..., A_p\}$, $C = diag\{C_1, ...C_p\}$ where $A_i \in \mathbb{R}^{\eta_i \times \eta_i}$ and $C_i \in \mathbb{R}^{1 \times \eta_i}$ are in Brunovsky canonical form.

- **H3**: There exist two sets of integer numbers $\{\sigma_1, ..., \sigma_n\}$ and $\{\theta_1, ..., \theta_p\}$ with $\theta_i > 0 \ \ \forall i$ such that:

  1. $\sigma_{\mu_i + l} = \sigma_{\mu_i + l - 1} + \theta_i$, $l = 1, ..., \eta_i - 1$; $i \in I_p = 1, 2, ..p$
  2. For $\chi = col\{\chi_1, ..., \chi_n\}$, $\sigma_i < \sigma_j \Rightarrow \frac{\partial \chi_i}{\partial x_j} \equiv 0$, $1 \leq i, j, \leq n$, $j \neq \mu_l$, $l \in I_p$

  where $\mu_1 = 1$, $\mu_i = \mu_{i-1} + \eta_{i-1}$, $i = 2, ..., p$

- **H4** : χ is locally Lipschitz with respect to w uniformly with respect to u.

Remark V.1 *Hypothesis* **H3** *is sufficient for the local uniform observability of system (1),* i.e. *that every input be universal for (1) (see García* et al.*, 2000 for details).*

A. The observer

The observer presented in (García *et al.*, 2000) is a high-gain one, and the parameter that controls such gain, γ, is assumed to have a fixed but arbitrary value. The observer is given by

- For $t \in I_k = [\tau_{k-1}, \tau_k)$ the prediction step is:

$$\begin{cases} \dot{\hat{w}} = A\hat{w}(t) + \chi(\hat{w}, u) \\ \dot{S}(t) = -S(t)\breve{Q}S(t) - A^T S(t) - S(t)A \\ \quad +C^T R^{-1} C \end{cases} \tag{14}$$

where $S = diag\{S_1, ..., S_p\}, S_i \in \mathrm{IR}^{\eta_i \times \eta_i}$.

- The correction step is: in $t = \tau_k$

$$\begin{cases} S(\tau_k^+) = S(\tau_k^-) + C^T R^{-1} C\delta \\ \hat{w}(\tau_k^+) = \hat{w}(\tau_k^-) \\ +S(\tau_k^+)^{-1} C^T R^{-1} \delta[y(\tau_k) - C\hat{w}(\tau_k^-)] \\ \hat{x}(\tau_k) = \phi^{-1}(\hat{w}(\tau_k^+)) \end{cases} \tag{15}$$

- The initial conditions $S_i(0) = S_{i_0}$ are symmetric positive definite matrices and $\hat{w}(0) = \phi(\hat{x}_0)$.

Here $\breve{Q}$ is a fixed positive definite and symmetric matrix, R is diagonal, and both matrices depend on γ (see (García *et al.*, 2000) for details). The following result, that was presented in that paper, establishes the convergence of the observer:

Theorem V.1 *If the conditions* **H2 - H4** *hold, there exists $\gamma_0 \in (0,1)$ such that for any $\gamma < \gamma_0$ if the sampling period δ is small enough, the system (14)-(15) is an observer for the nonlinear system (1) (for the estimation $\hat{x}(t) = \phi^{-1}(\hat{w}(t))$, that verifies:*
a) when the measures are noiseless, it is an exponential observer.
b) For noisy measurements, the estimation error variance is bounded, and the bound is proportional to the noise variance.

B. The algorithm

The combination of the observer and the Algorithm 1 gives origin to the *Output feedback algorithm for trajectory tracking,* Algorithm 2, that for a positive number T, a sampling period δ and a boundary layer thickness ϵ may be described as follows.

1. In the interval $[T_N(\delta), T_{N+1}(\delta))$ we solve the initial value problem (5).
2. For $T_N(\delta) \le \tau_k \le t < \tau_{k+1} \le T_{N+1}(\delta)$, the control $u(t)$ that we apply *to the plant and to the observer* is given by (9) with b_j given by (11) with $\hat{x}(\tau_k)$ given by (15) and $z(\tau_{k+1})$ as in (10).

The next result, whose proof we omit, assures that for small enough initial estimation error and adequate sampling periods, Algorithm 2 will work properly.

Theorem V.2 *Let x^*, $\omega(t,x)$, R_0 and ε_0 as in Theorem III.1, and suppose that system (1) verifies* **H2 - H4**. *Then there exist positive numbers α, T, δ_0, ϵ, $\hat{d}^*$ and T' and a number $\gamma_0 \in (0,1)$ such that if the observer gain $\gamma < \gamma_0$, the sampling rate $0 < \delta \le \delta_0$ and $x(\cdot)$ is a trajectory of (1) corresponding to the control $u(\cdot)$ given by Algorithm 2 such that $\|x(0) - \hat{x}(0)\| < \hat{d}^*$, and $\|x(0) - x^*(0)\| \le R_0$, we have:*

1. *there exist a $\mathcal{K}$-class function Δ, depending on both β and γ such that $\|x(t) - x^*(t)\| \le \Delta(R_0 + \varepsilon_0) + \varepsilon_0 \quad \forall t \ge 0$*
2. *$\|x(t) - x^*(t)\| \le \varepsilon_0$ if $t \ge T'$.*

Remark V.2 *If in addition ϵ is not too small the control $u(\cdot)$ so obtained presents no chattering effect.*

C. Simulation

In this section we apply Algorithm 2 to the system of the example given in section III. In order to do so we adopt the output function $h(x) = x_1$. Since now the system can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \chi_1(x,u) \\ \chi_2(x,u) \end{bmatrix}$$

$$y = [1, 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with $\chi_1(x,u) = u_1(1 - x_1^2)$, $\chi_2(x,u) = -x_1 + u_2(1 - x_2^2)$, the hypotheses **H2 - H4** are satisfied with $\{\sigma_1, \sigma_2\} = \{1, 2\}$, $\eta_1 = 2$, $\mu_1 = 1$ and $\theta_1 = 1$.

For the simulations we consider a (more realistic) noisy output measurement process with normal distribution given by $N(0, 0.01)$, *i. e.* $y(\tau_k) = x_1(\tau_k) + \nu_k$ with $\mathcal{E}(\nu_k \nu_s) = \delta_k^s 0.01$.

The design parameters adopted are: $\hat{d}^* = 0.22$, $\gamma = 0.7$, $\breve{Q}_1 = diag\{2.08, 4.25\}$, $S_i(0) = S_{i_0} = I$ where I is the identity matrix, and initial conditions for the observer $(\hat{x}_1(0), \hat{x}_2(0)) = (0.4, 0)$. The boundary layer thickness was taken as $\epsilon = 0.1$, and the other parameters were taken as in the simulation in section III.

The simulation results are shown in Figs. 3 - 6. In Fig. 3 the estimation errors $x_1(t) - \hat{x}_1(t)$ and $x_2(t) - \hat{x}_2(t)$ are shown. In Fig. 4 the tracking errors $|e_r(t)|$ (dotted) and $|e_{ri}(t)|$ are exhibited, being $e_{ri}(t)$ the ideal tracking error of system (8), controlled by the *CLU* $\omega(t,x)$. Figure 5 exhibits the tracking controls, as given by Algorithm 2. It can be seen that no chattering effect is present. Figure 6 shows the tracking controls also given by Algorithm 2, but in the case on noiseless measurements. This figure shows that the blurring effect that appears in the controls in Fig. 5 is due to the noise.

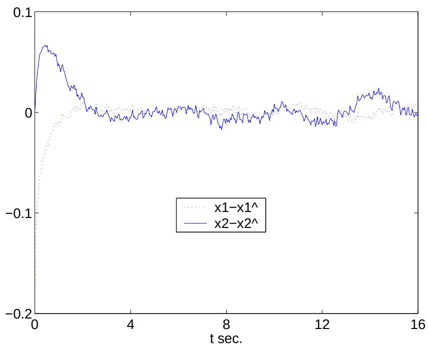

Figure 3: Algorithm 2: estimation errors.

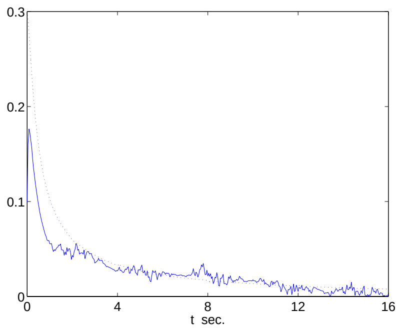

Figure 4: Algorithm 2: tracking errors.

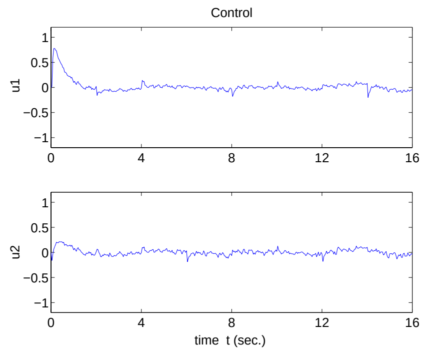

Figure 5: Algorithm 2: controls

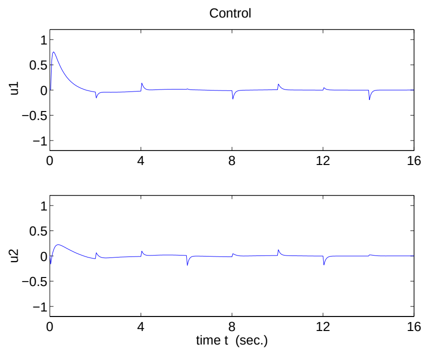

Figure 6: Algorithm 2: noiseless controls

The final tracking error values, as obtained in this simulation and in that of section III, and the behavior of the controls, are presented in the following table. The second column of the table shows $|e_r(t)|$ for $t \geq 8(sec.)$ and the third one, the control behavior.

Algorithm	$\|e_r(t)\| \leq$	Control u:
Algorithm 0	0.063	chattering
Algorithm 2	0.018	No chattering

VI. CONCLUSIONS

In this work we presented an algorithm that improves the one of (Mancilla Aguilar *et al.*, 2000a) for the digital implementation of trajectory tracking controllers for control affine systems. This algorithm reduces the chattering effect that appears in the original algorithm while keeping a good tracking error performance. An observer was introduced in order to cope with the case where the states were not available, providing an estimation of the states. It was shown to work properly in a closed loop scheme even with noisy output measurements. Simulations were presented that exhibit the improvements of this strategy for an affine control system.

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