NONLINEAR H_∞ CONTROL OF AN EXPERIMENTAL pH NEUTRALIZATION SYSTEM

L.G.S. LONGHI†, E.L. LIMA‡, P.R. BARRERA# and A.R. SECCHI#

† *Depto. de Eng. Química, PUCRS, Av. Ipiranga, 6681, prédio 30, 90619-900, Porto Alegre, RS, Brasil*
llonghi@pucrs.br
‡ *Programa de Engenharia Química, COPPE/UFRJ, Rio de Janeiro, RJ, Brasil*
enrique@peq.coppe.ufrj.br
Depto. de Eng. Química, UFRGS, campus central, Porto Alegre, RS, Brasil
{barrera, arge)@enq.ufrgs.br

Abstract–– **The classical control theory is based on the design of linear controllers for systems described by linear models. However, there exist some situations where it is not recommended, or even impossible, to use a linear controller. One of those situations arises when the magnitude of the process gain experiences a dramatic variation within the operating range of interest. A classic example of a chemical process where this situation occurs is the pH control around the neutralization point in a continuous stirred tank. In this work, the pH control for a strong acid – strong base system is addressed. To solve this problem, a nonlinear H_∞ control law is derived based on a nonlinear model previously developed. The attainment of that control law is done with the help of recent mathematical results from the authors concerning the solution of Hamilton-Jacobi-Isaacs inequalities. The nonlinear controller is implemented on an experimental reactor and its performance is compared with a PID control law tuned according to the classical minimum error integral criteria. The obtained results show that the nonlinear H_∞ control theory can be a good alternative to solve this difficult SISO (Single Input – Single Output) control problem.**

Keywords–– **Nonlinear Control, pH control, Nonlinear H-infinity Control.**

I. INTRODUCTION

The theory of classic control is based on the design of linear controllers for systems described by linear models or nonlinear models linearized around an operating point. However, there exist some situations where it is not recommended to use a linear controller. One of those situations arises when the magnitude of the process gain experiences a dramatic variation within the operating range of interest. In this situation, the use of a fixed linear controller can lead to a poor performance of the closed loop system and even to its loss of stability. A classical example of a chemical process where this situation happens is the pH control around the neutralization point in a continuous stirred tank. In this control problem, the titration curve – which represents the system's input-output mapping – presents a highly nonlinear behavior in response to addition of acid or base. This behavior is amplified even more if the reagents are strong acid and/or base.

In the present work, the objective is to control the pH of an experimental system within this difficult range of operation. The problem can be stated as to maintain the pH in the neutralization point manipulating a strong base stream flow rate in response to disturbances on the strong acid flow rate. To solve this disturbance attenuation problem, a nonlinear H_∞ controller is designed and implemented in a bench-scale plant. This synthesis approach is possible due to recent mathematical results from the authors concerning the solution of Hamilton-Jacobi-Isaacs inequalities (Longhi et al., 2000). It must be emphasized that this control law synthesis is not based on any kind of linearization procedure, such as multi-linear models (Galán et al., 2000), gain scheduling or adaptive schemes (Sung et al., 1998), nor in a change of control objective to fit a known solution method (Li and Zhang, 1999).

In the sequence of this work, the relations between the disturbance attenuation problem and the nonlinear H_∞ control theory are briefly explained in section 2. In section 3, the experimental apparatus is described and a mathematical phenomenological model is developed and compared to the experimental data. The controller synthesis and the experimental results are presented in section 4. Finally, in section 5, the conclusions are presented.

II. DISTURBANCE ATTENUATION AND NONLINEAR H_∞ CONTROL THEORY

Consider the IA (Input-Affine) nonlinear system description of Eq. (1).

$$\dot{x} = f(x) + g(x)u + k(x)w$$
$$z = \begin{bmatrix} h(x) \\ u \end{bmatrix} \tag{1}$$

where $x \in M$ ($M \subseteq \Re^n$) is the vector of the system's state variables defined on a neighborhood of the origin, $w \in \Re^q$ is the vector of exogenous inputs, $u \in \Re^m$ is the vector of control inputs, and $z \in \Re^s$ is the vector of exogenous outputs which characterizes the control objective. The mappings f(x), g(x), k(x) and h(x) are assumed to be nonlinear smooth functions and, for simplicity, f(0) = h(0) = 0.

The disturbance attenuation problem via state feedback is concerned to the construction of a feedback controller, u(x), satisfying two objectives: (1) To asymptotically stabilize the resulting closed-loop plant, and (2) To minimize the influence of the exogenous inputs, w, on the objective variable, z. If the influence from w(t) on z(t) is measured as the finite L_2-gain between these variables, the disturbance attenuation problem can be solved by using the results from the nonlinear H_∞ control theory (Isidori and Astolfi, 1992). Here, the L_2-gain is defined as in Van der Schaft (1992).

Definition 1 (Finite L_2-gain). Given any $\gamma > 0$, the mapping from w(t) to z(t) is said to have finite L_2-gain less than or equal to γ if, under the zero initial condition $x(0) = 0$,

$$\int_0^T \|z(t)\|^2 dt \le \gamma^2 \int_0^T \|w(t)\|^2 dt \qquad (2)$$

for all T ≥ 0 and all w(.) ∈ $L_2(0,T)$, where || • || denotes the Euclidean norm, and $L_2(0,T)$ denotes a Hilbert space composed by all real variable functions, Lebesgue-measurable, defined on the interval [0, T].

The finite L_2-gain can be viewed simply as the maximum amplification, for every time T > 0, on the variable z(t), measured in terms of its energy (Euclidean) norm, caused by a energy-limited external input w(t). For a linear SISO system, the finite L_2-gain turns back to be the usual system open-loop gain.

However, as the minimization of the L_2-gain can lead to a controller with a very small validity region (Yazdanpanah et al., 1999) or very near to the stability frontier (Keel and Bhatacharyya, 1997), it is usual in the literature to consider only suboptimal solutions to the problem. In this case, the minimum of γ is replaced by the gain attenuation at some acceptable level. The suboptimal solution to the nonlinear H_∞ control problem for a system described by Eq. (1) can be given by theorem 1 (Van der Schaft, 1999).

Theorem 1 (Local sub-optimal solution to the nonlinear H_∞ control problem via state feedback for IA systems). *Consider the nonlinear system of Eq. (1) and a real parameter $\gamma > 0$. Suppose that exists a smooth positive definite solution, V(x) > 0, to the HJI (Hamilton-Jacobi-Isaacs) inequality given by Eq. (3),*

$$H_*(x) = \frac{\partial V}{\partial x}(x) f(x) + h^T(x) h(x)$$
$$- \frac{1}{4}\frac{\partial V}{\partial x}(x)\left(g(x) g^T(x) - \frac{1}{\gamma^2} k(x) k^T(x) \right)\frac{\partial^T V}{\partial x}(x) < 0 \qquad (3)$$

then, the closed-loop system with the feedback control law of Eq. (4),

$$u_*(x) = -\frac{1}{2} g^T(x) \frac{\partial^T V}{\partial x}(x) \qquad (4)$$

is asymptotically stable at the origin and has locally a L_2-gain (from w to z) less or equal to γ. Moreover, the worst-case disturbance is given by Eq. (5).

$$w_*(x) = \frac{1}{2\gamma^2} k^T(x) \frac{\partial^T V}{\partial x}(x) \qquad (5)$$

It must be noted that Theorem 1 does not give a method to solve the problem nor furnishes the size of the local state-space region where its solution works. In fact, these are the main limitations to apply the results from the nonlinear H_∞ theory to real systems. Before to state the developed solution, it is necessary to define which is the validity region for the nonlinear H_∞ controller. This definition was based on (Yazdanpanah et al., 1999).

Definition 2 (Nonlinear H-infinity controller validity region). The region of the state space of Eq. (1) that, subject to the nonlinear state feedback law from theorem 1, simultaneously satisfies the HJI inequality and guarantees asymptotic stability of the worst-case disturbance of Eq. (5) in closed-loop system, is referred to as the validity region corresponding to the controller of Eq. (4). Any region that is a subset of this state-space region is referred to as an estimate of the validity region.

In this work, the solution for the nonlinear H_∞ control problem is found by solving the optimization problem 1 (Longhi et al., 2000) shown later on. This optimization problem, based on preliminary results concerning the positivity of multivariable scalar functions (Longhi et al., 2001), requires the definition 3 and its solution furnishes a control law associated with a validity region. For more details, it is recommended to read the references of this paragraph.

Definition 3 (Real local region). The real local region of a multivariable scalar function y(x) is the set composed by the subsets of the real field where each element of x can assume values such that y(x) is real and y ≠ 0 unless x = 0.

Optimization problem 1 (Nonlinear H_∞ control maximizing the size of the validity region). *Choose the form of function* V(x) *and substitute in* $H_*(x)$. *Write these two functions as quadratic form representations:* $V(x) = \Theta(x)^T P_V \Theta(x)$ *and* $H_*(x) = \Phi(x)^T P_H \Phi(x)$, *where* P_V *and* P_H *are symmetric real matrices obtained directly from the coefficients of* V(x) *and* $H_*(x)$, *respectively. Write the time derivative of* V(x) *as the quadratic form representation:* $\dot V(x) = \Psi(x)^T P_{Vd} \Psi(x)$, *where* P_{Vd} *is a symmetric real matrix obtained directly from the coefficients of* $\dot V(x)$. *Let* α_i , β_i *and* δ_i *be the parameters which define the positivity region of* V(x) *and the nega-*

tivity regions of $H_*(x)$ *and* $\dot{V}(x)$*, respectively. Choose the parameters of* $V(x)$, $\Theta(x)$, $\Phi(x)$ *and* $\Psi(x)$ *in a way to maximize the region defined by* $V(x) = C$ *subject to the constraints* $P_V > 0$, $P_H < 0$, $P_{Vd} < 0$ and $\gamma_{min} < \gamma < \gamma_{max}$. *The parameter* $C \in \Re^+$ *is obtained as the minimum value of* $V(x)$ *intersecting the positivity of* $V(x)$ *and the negativity region of* $\dot{V}(x)$ *and* $H_*(x)$. *The solution,* $V(x)$, *solves locally the problem within the validity region defined by* $V(x) < C$ *for the* γ*–level attenuation.*

Roughly speaking, the optimization problem 1 tries to find a solution to the HJI inequality that maximizes the size of the validity region associated with that solution. This is quite different from the usual approach in the nonlinear H_∞ control theory where the main objective is to find a local solution that minimizes γ regardless to the fact that the resulting controller has a practical validity region or not. Often, this usual procedure leads to a very fragile controller.

Despite the fact that optimization problem 1 can be considered a very general approach to solve the nonlinear H_∞ control problem via state feedback, usually it is a very complex one, many times intractable. To reduce its dimension, some simplifications can be done. One expected problem occurs when it is desired to use non-ellipsoidal (non-quadratic) forms to represent V(x). In these cases, it could be very tedious to find an equation for the area of V(x) = C. Furthermore, the resulting equation can be very complex, thus inadequate for using in an optimization problem. So, aiming the simplification of the problem, it is recommended to use, when possible, quadratic forms to represent V(x).

Moreover, if the situations 1 and/or 2 below occur, the size of the problem can be considerably reduced:

1. If the condition shown in Eq. (6) is satisfied, then the signal of $\dot{V}(x)$ does not need to be evaluated.

$$\left(g(x)^T g(x) - \frac{1}{\gamma^2} k(x)^T k(x) \right) \geq 0 \tag{6}$$

In this case, the HJI inequality satisfaction implies the negativity of $\dot{V}(x)$ in the same state space region. This situation occurs when the description of the IA system is known and the lower bound of γ is defined as the minimum necessary to assure that the inequality (6) holds.

2. If V(x) is globally positive definite, like quadratic forms, for example, then the parameters α_i are eliminated from the optimization problem.

An additional relationship among the theorem 1 and the optimization problem 1 solved for a quadratic V(x), as well as a sketch of the proof for the claim of Eq. (6), are found in appendix A.

III. NEUTRALIZATION SYSTEM MODELING

The experimental apparatus considered in this work is a reactor (a two-liters glass reactor) and pH, temperature and flow sensors, coupled with a control and monitoring unit. The reactor operates at atmospheric pressure and environment temperature, being continuously stirred at the 500 rpm. The reactor vessel is fed with two input streams provided with peristaltic pumps. The acid stream contains 0.1 M HCl, and the basic stream 0.1 M NaOH. The amount of fluid given by the sum of these streams is removed from the reactor. As a consequence, the reactor behaves like a typical CSTR (Continuous Stirred Tank Reactor) with constant volume. In this work, the acid stream flow rate is considered as the disturbance variable and the base stream flow rate as the manipulated variable. A sketch of that control system is shown in Fig.1.

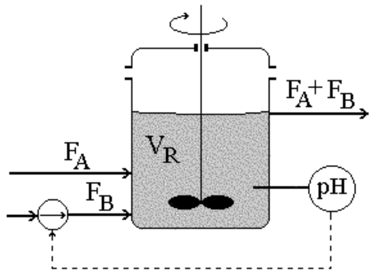

Figure 1. The continuous pH neutralization process scheme.

The process model used in this work considers the change of coordinate proposed by (Narayanan et al., 1998): $\eta = [H^+] - [OH^-]$. The relation between the variable η and the original variable (pH) is given by Eq. (7).

$$\eta = 10^{-pH} - \frac{K_W}{10^{-pH}} \Leftrightarrow pH = -log_{10}\left(\frac{\eta + \sqrt{\eta^2 + 4K_W}}{2} \right) \tag{7}$$

The main advantage of using η instead of pH is the attainment of a concise IA process model, suitable for the synthesis of the disturbance attenuation controller, as it can be seen in Eq. (8).

$$\frac{1}{V_R}\frac{d\eta}{dt} = F_A C_{A0} - F_B C_{B0} - (F_A + F_B)\eta \tag{8}$$

where V_R is the reactor volume, F_A is the acid stream flow rate, F_B is the basic stream flow rate, C_{A0} is the acid concentration in the acid stream, C_{B0} is the base concentration in the basic stream and $K_W = 10^{-14}$ is the equilibrium constant of water dissociation.

Fig. 2 presents the comparison between the phenomenological model of Eqs. (7)-(8) and the experimental data. The highly nonlinear behavior of the pH system can be easily seen in this figure. The modeling error in the basic region, due to the acid characteristic of the available water used in the experiments (pH ranging from 5 to 7), can also be seen in this figure.

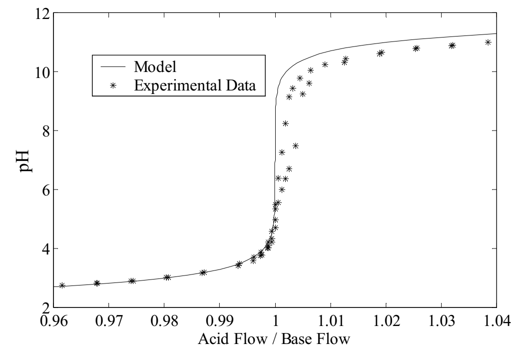

Figure 2. Comparison between the model of Eq. (8) and experimental data.

IV. NONLINEAR CONTROL SYNTHESIS AND EXPERIMENTAL RESULTS

In order to obtain a description whose steady state of interest is the origin, the following new variables were defined: $\theta_A = F_A / V_R$, $\theta_B = F_B / V_R$, $x_1 = \eta - \eta_{SS}$, $w = \theta_A - \theta_{ASS}$ and $\nu = \theta_B - \theta_{BSS}$. The sub-index SS denotes the steady state values for the variables. For the neutral pH considered in this work, the values of these variables are: $\eta_{SS} = 0$ and $\theta_{BSS} = \theta_{ASS} = 0$.

Furthermore, to assure the off-set elimination, an additional state was incorporated to the original control variable: $\nu(x) = u(x) + T_i\, x_2$, where T_i is a parameter to be chosen and the new state, x_2, has the dynamics of an integrator: $\dot{x}_2 = x_1$. So, if instead of u(x), it is implemented the new control law ν(x), then the term $T_i\, x_2$ can be viewed as an integral mode, being sufficient to eliminate the persistent deviations generated by modeling errors and non-vanishing perturbations. This is important because the nonlinear H_∞ control theory was originally developed to deal only with vanishing disturbances.

Now, the control system can be adequately represented by Eq. (9) and the objective variable $z = \begin{bmatrix} h(x) \\ u \end{bmatrix} = \begin{bmatrix} x_1 \\ u \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\dot{x} = \begin{pmatrix} -(C_{B0} + x_1) T_i\, x_2 \\ x_1 \end{pmatrix} + \begin{pmatrix} -(C_{B0} + x_1) \\ 0 \end{pmatrix} u + \begin{pmatrix} C_{A0} - x_1 \\ 0 \end{pmatrix} w \quad (9)$$

If a simple Lyapunov function $V(x) = a x_1^2 + b x_2^2$ is considered, the HJI inequality (10) must be solved for $V(x) > 0$.

$$\begin{aligned} H_*(x) = & \left(\frac{a^2}{\gamma^2} - a^2 \right) x_1^4 + \left(-0.2 \frac{a^2}{\gamma^2} - 0.2 a^2 \right) x_1^3 \\ & - 2 a T_i x_1^2 x_2 + \left(-0.01 a^2 + 0.01 \frac{a^2}{\gamma^2} + 1 \right) x_1^2 \\ & + \left(-0.2 a T_i + 2b \right) x_1 x_2 < 0 \end{aligned} \quad (10)$$

As the LaSalle's detectability condition is satisfied, it is sufficient to find a solution for the non-strict inequality $H_*(x) \le 0$ to solve the disturbance attenuation problem. In order to cancel the quadratic cross product ($x_1 x_2$) from inequality (10), it is assumed that T_i = 10 b/a. So, the HJI inequality can be rewritten as:

$$\begin{aligned} H_*(x) = & a^2 \lambda\, x_1^4 - 0.2\, a^2 (\lambda + 1) x_1^3 \\ & - 20\, b\, x_1^2 x + (1 + 0.01 a^2 \lambda) x_1^2 < 0 \end{aligned} \quad (11)$$

where $\lambda = \left(\frac{1}{\gamma^2} - 1 \right)$.

A necessary condition for a local solution to inequality (11) is $(1 + 0.01 a^2 \lambda) \le 0$. This situation only occurs if $\lambda < 0$. This implies, equivalently, $\gamma > 1$. Then, the lower bound of γ was fixed at 1 in the formulation of the optimization problem 1. In addition, as a consequence of the non-negativity of the perturbations, the condition of Eq. (6) is automatically satisfied, and the searching for a local solution to inequality (11) becomes easier.

To solve the optimization problem 1, the quadratic form representation of Eq. (12) was considered.

$$H_*(x) = \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ \sqrt{\beta_1 + x_1 x_1} \\ \sqrt{\beta_2 + x_2 x_1} \end{bmatrix}^T . P_{H*} . \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ \sqrt{\beta_1 + x_1 x_1} \\ \sqrt{\beta_2 + x_2 x_1} \end{bmatrix} \quad (12)$$

where the Matrix P_{H*} is given by:

$$\begin{bmatrix} \begin{matrix} 1 + 0.01 a^2 \lambda + 0.2 a^2 (\lambda + 2) \beta_1 \\ + 20 b \beta_2 \end{matrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^2 \lambda & 0 & 0 \\ 0 & 0 & 0 & -0.2 a^2 (\lambda + 2) & 0 \\ 0 & 0 & 0 & 0 & -20b \end{bmatrix}$$

and the parameters β_1 e β_2 are two positive numbers that control the size of the negativity region of $H_*(x)$.

So, the optimization problem 1 can be stated according to Eq. (13).

$$[a, b, \beta_1, \beta_2, \gamma] = \underset{a,b>0}{arg\ min} (a.b) \quad (13)$$
$$P_{H*} \le 0$$
$$\beta_1, \beta_2 > 0$$
$$\gamma_{min} < \gamma < \gamma_{max}$$

In order to simplify the shape of the controller validity region, it was assumed that $\beta_1 = \beta_2$. In addition, it was arbitrarily chosen that the maximum and minimum levels of attenuation (γ) are 1.0 and 1.5, respectively. The solution for the problem of Eq. (13) is given by:

$$\begin{cases} a = b = 2837.1 \\ \beta_1 = \beta_2 = 1.874\,x\,10^{-2} \\ \gamma = 1.5 \\ T_i = 10 \end{cases} \quad (14)$$

Finally, from the sub-optimal control law, $u_*(x) = -\frac{1}{2} g^T(x) \frac{\partial^T V}{\partial x}(x) = a(CB_0 + x_1)x_1$, one can construct the nonlinear H∞ controller, $v(x) = u_*(x) + T_i\,x_2 = 2837.1(0.1 + x_1)x_1 + 10x_2$, as a function of the original control variable η: $F_B(\eta) = v(x)V_R$. This results in Eq. (15).

$$\begin{cases} F_B(\eta) = 1.5\,x\,10^3\left(283.71(\eta - \eta_{SP}) + 2837.1(\eta - \eta_{SP})^2 + 10\,x_2\right) \\ \dot{x}_2 = (\eta - \eta_{SP}) \end{cases} \quad (15)$$

This control law has a validity region given by: $V(x) = 2837.1(x_1^2 + x_2^2) < 1$. This validity region is equivalent to the pH region given by: 2 < pH < 12. Since the disturbance attenuation problem solved by the nonlinear H∞ control theory only considers vanishing perturbations, for the more realistic case of persistent disturbances, it is hoped that the controller validity region is sufficiently big to support large disturbances on w(t).

To implement the controller of Eq. (15) in the experimental plant described in section 3, it was developed an interface to connect the plant to a remote computer. This interface was written using the Matlab/Simulink environment.

In Fig. 3, the nonlinear H∞ controller performance is shown in response to some disturbances on the acid stream (see Table 1). In the same figure, it is plotted the performance of a usual PI controller tuned according to the classical minimum integral error criteria. The PI tuning was made using the ITAE parameters for the disturbance attenuation case and an experimentally identified FOPDT (First Order Plus Dead Time) model (Seborg et al., 1989). More details concerning the model identification and the PI tuning are found in Appendix B.

Table 1 - Disturbance applied on the acid stream

Time [s]	Acid flow rate [ml/min]
0	0
0^+	2.40
500	3.52
1000	4.80
1300	1.28

In Figure 4, the responses of both controllers are shown for set-point changes. It can be noted that the good properties of the nonlinear H∞ controller do not hold. In fact, as distant from the neutralization point, worst is the nonlinear H∞ controller performance. One should suspect that this is because the nonlinear controller was not designed to face the set-point tracking problem. However, this can be refused because dynamic simulations, not shown here, for the closed-loop system present the same good behavior for different set-points. This fact makes clear that there are other sources for the weak controller performance at high pH values, one of these being the model mismatch at basic pH values.

However, not only the model mismatch affects the performance in those conditions but, mainly, the input constraints (magnitude and speed saturation) also worsen the performance at high pH values. This is an experimental drawback that does not occur in the simulations. This could be explained because the control law was obtained without considering the input saturation. Thus, the control law will only have a valid association with the controller within a range where these constraints are not violated. Regarding the experimental system of this work, this valid region is constrained to pH values ranging from approximately 6 to 8.

Furthermore, because of the input saturation (in magnitude) it is not possible any kind of implementation for pH above 9.5 and below 4.5. In these regions, other more conservative control laws should be used. This is why no experimental tests for pH higher than 9 or lower than 5 were considered. A possible ad-hoc solution for this problem could be found by choosing different controller's settings for each operating condition, characterizing a gain scheduling procedure. However, this approach was not considered here because it implies loss of optimality.

V. CONCLUSIONS

The pH control of a continuous stirred tank by a disturbance attenuating controller using the nonlinear H∞ control theory results was presented. Based on previous results from the authors concerning the solution of Hamilton-Jacobi-Isaacs inequalities, a synthesis approach to solve quantitatively the nonlinear H∞ control was developed. A quantitative solution means a control law associated with its validity region. The validity region is the state-space region where the stability and performance requirements are satisfied. To solve the problem realistically, instead of looking for the best possible performance, the developed procedure aims the maximization of the validity region while guaranteeing a minimal performance. This is important because the optimal controller is usually a fragile one. Some alternatives to simplify the optimization problem were also discussed. The developed methodology was applied to control a strong acid - strong base pH continuous system using the base stream flow rate as the manipulated variable and the acid stream flow rate as the disturbance variable. To make the process model an input-affine one, a change of variable was performed. The resulting nonlinear controller was implemented in an experimental two-liter bench-scale plant and its responses for disturbances on the acid stream flow rate and set-point changes were compared to a well-tuned PI controller subject to the same operating conditions.

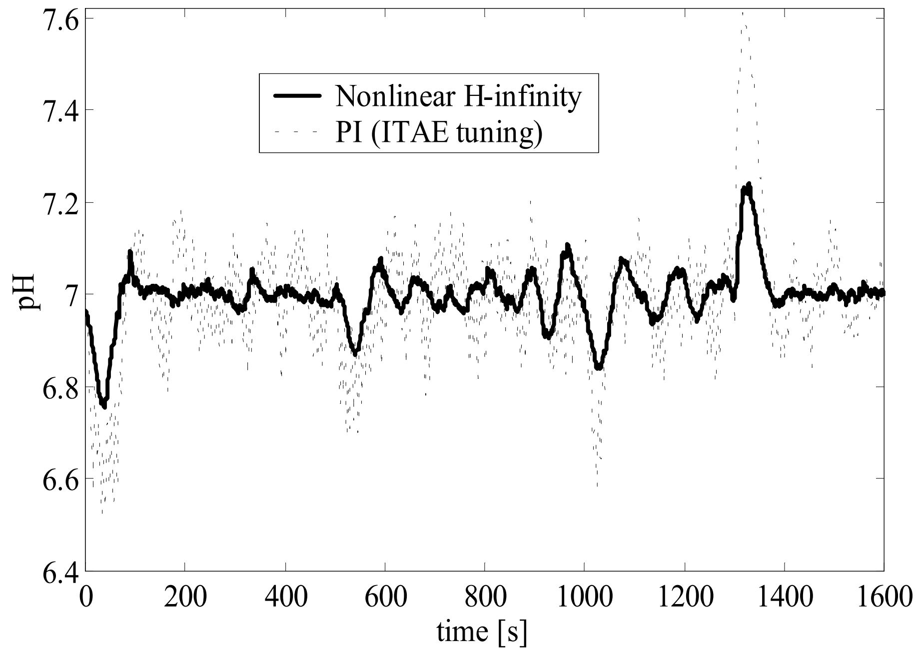

Figure 3 - Performance comparison between the nonlinear H_∞ controller and an ITAE-tuned PI controller for disturbance attenuation.

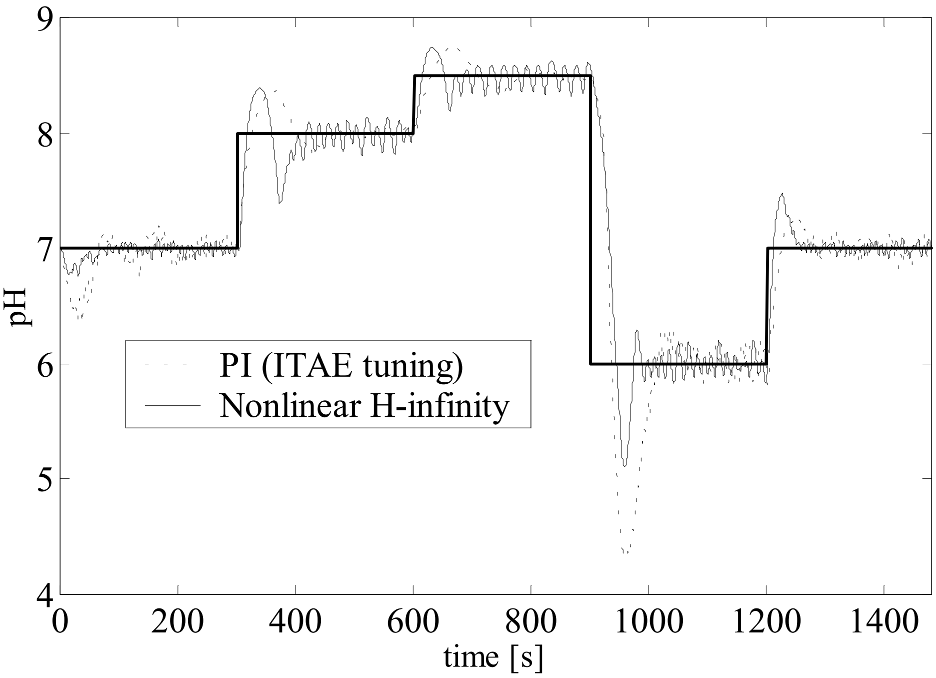

Figure 4- Comparison between the nonlinear H_∞ controller and the PI controller for set-point changes. (The solid line steps represent the set-points)

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APPENDIX A: OPTIMIZATION PROBLEM 1 SOLVED FOR A QUADRATIC FORM LYAPUNOV FUNCTION

This section shows the relation between the solution of the optimization problem 1, solved for a quadratic form, and the theorem 1. To make this relation clear, the optimization problem is simplified and applied to design a disturbance attenuation controller for an IA system.

Consider the IA nonlinear system of Eq. (1) and the optimization problem 1 solved for a positive definite function V(x). If this Lyapunov function is considered to be a quadratic form, $V(x) = x^T P x$, the solution of the optimization problem will furnish not only the elements of the matrix P and the γ-level attenuation, but an associated ellipsoidal estimate for the DA (Domain of Attraction), given by $V(x) = x^T P x < C$, where $C \in \Re^+$. According to the theory of quadratic forms (Lam, 1973), if V(x) is a homogeneous positive definite form, it will be not only locally positive but globally, too. Thus, as pointed in section 3, there is no need for choosing a parameter to control the size of its positivity region. Only the Sylvester's criteria conditions for positivity should be appended to the inequality constraints set.

Besides this simplification, by noting that the HJI inequality from theorem 1, given by Eq. (3), can be rewritten according to Eq. (16),

$$\begin{aligned} H_*(x) &= V_x\big(f(x)+g(x)u_*(x)+k(x)w_*(x)\big) \\ &+ h^T(x)h(x) + \left\| \left(-\frac{1}{2}g(x)^T V_x(x)^T\right)\right\|^2 \\ &-\gamma^2\left\|\left(\frac{1}{2\gamma^2}k(x)^T V_x(x)^T\right)\right\|^2 < 0 \end{aligned} \quad (16)$$

it can be observed that the first term of the right side is the time derivative of V(x) for IA systems fed with the optimal feedback, $u_*(x)$, and perturbed by the worst disturbance, $w_*(x)$.

Consequently, if this time derivative is denoted by $\dot V(x) = V_x\big(f(x)+g(x)u_*(x)+k(x)w_*(x)\big)$, Eq. (16) can be rewritten as Eq. (17):

$$\begin{aligned} \dot V(x) &< -h(x)^2 \\ &-\frac{1}{4}V_x(x)^T\left(g(x)^T g(x) - \frac{1}{\gamma^2}k(x)^T k(x)\right)V_x(x) \end{aligned} \quad (17)$$

Now, it is easy to see that, if $\left(g(x)^T g(x) - \frac{1}{\gamma^2}k(x)^T k(x)\right) \geq 0$, then the HJI inequality, $H_*(x) < 0$, implies $\dot V(x) < 0$ in the same state space region. As a consequence, the optimization problem 1 is greatly simplified, because it is not necessary anymore to verify the negativity of the time derivative to assure the asymptotic stability for the worst case disturbance, $w_*(t)$.

Now, the resulting optimization problem is reduced to the choice of the quadratic form representation for the HJI inequality containing a parameter set (β_i) which controls the size of its negativity region. Besides the β_i parameters, before going to the solution of the problem itself, it is necessary to find some extra conditions constraining the size of the validity region to be within a feasible region. This equality constraint can be written according to V(x) = C , where $C \in \Re^+$ can be obtained by solving the equation $H_*(x) = H_*(x, \beta_i) = 0$ for x and substituting the obtained values, $x = x(\beta_i)$, in V(x).

After a suitable choice for the quadratic form representation, the optimization problem searches for a matrix P (from the quadratic form $V(x) = x^T P x$) and the γ-level of attenuation. If a solution ($P > 0$ and $\gamma > 0$) for the remaining problem is found, according to theorem 1, within its associated ellipsoidal validity region, V(x) = C, both performance – given by $V(x) \geq 0$ and $H_*(x) < 0$ – and asymptotic stability (given by the domain of attraction – the region where $V(x) > 0$ and $\dot{V}(x) < 0$) requirements are simultaneously satisfied.

As pointed out in section 2, the main difference between the approach of optimization problem 1 and the usual approach to solve the nonlinear H_∞ control problem is that, instead of trying to find a local solution that minimizes γ, the former one tries to find a solution that maximizes the controller validity region. Despite the simplifications to the optimization problem 1, the resulting problem is still a nonconvex one, being necessary to use global optimization techniques to find less conservative solutions.

APPENDIX B: FOPDT MODEL IDENTIFICATION AND PI-ITAE TUNING

As the experimental pH system was designed to work near to neutral values, the PI controller was tuned to work well at the same conditions. Thus, it was considered a PRC (Process Reaction Curve) approach to identify the process parameters of a FOPDT model near to this range. Thus, a 10 seconds pulse on the acid stream at its maximum flow rate (8 mL/s ≈ 1.33 x 10^{-3} L/s) was implemented and the pH response was plotted from an initial value of 7.8 until the new steady state be reached. As the acid and base stream are both 0.1 M and have the same dissociation constant, the acid pulse effect is equivalent to a base pulse with the same duration. This plot is shown in Figure 5. Then, the usual hand-made procedure, as described in Seborg et al. (1989) and other classical control books, was performed. The FOPDT model approximation found is presented in Eq. (19).

$$G(s) = \frac{K\,e^{-\theta s}}{\tau s + 1} = \frac{577\,e^{-5s}}{180\,s + 1} \qquad (19)$$

where K represents the gain (pH units per liters of acid), τ is the first order time constant (seconds), and θ is the dead-time (seconds). Based on these parameters, the PI parameters were computed according to the classical minimum integral error criteria for load rejection (Seborg et al., 1989), Eqs.(20)-(21).

$$K_C = \frac{0.859}{K}\left(\frac{\theta}{\tau}\right)^{-0.977} = 0.0493 \approx 0.05\,[L_{base}\,/\,pH] \qquad (20)$$

$$\tau_I = \frac{\tau}{0.674}\left(\frac{\theta}{\tau}\right)^{0.680} = 23.35 \approx 23\,[s] \qquad (21)$$

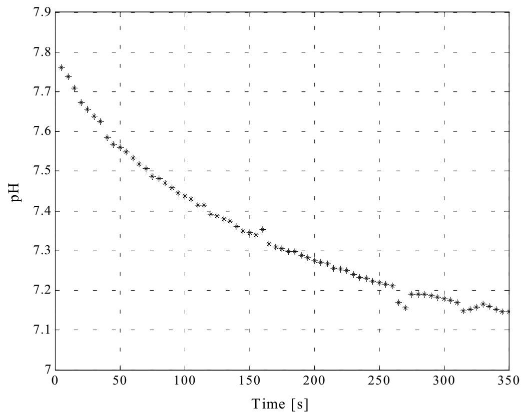

Figure 5. Response of the experimental pH system for a 10 seconds pulse on the acid stream.

The resulting PI controller, Eq. (22), was implemented and compared to the disturbance attenuation controller of section 4 for the same experimental situations.

$$\begin{cases} V_B(pH) = 0.05\left[(pH - pH_{SP}) + \frac{1}{23}x_2\right] \\ \dot{x}_2 = (pH - pH_{SP}) \end{cases} \qquad (22)$$

Note that the huge difference among the gain of the controller of Eq. (22) and Eq.(15) is due to the different magnitude between pH and η. Also observe that equation set (22) furnishes the amount of base to be fed inside the reactor on each sample time by the peristaltic pump. The base stream flow rate, $F_B(t)$, may be obtained by dividing $V_B(pH)$ by 5 seconds (the sample time used in the experimental coupled unit).

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