AN OBSERVER FOR CONTROLLED LIPSCHITZ CONTINUOUS SYSTEMS

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***Abstract*— In this paper we present an observer for controlled nonlinear systems that are locally Lipschitz continuous in both the state and control variables. This observer is based on a recently introduced model of observer for autonomous Lypschitz continuous systems, and can be designed to realize an arbitrary, finite accuracy when both the state space and control variables evolve in bounded regions.**

***Keywords*— Nonlinear observers, Lipschitz continuous systems.**

I. INTRODUCTION

The design of observers for nonlinear systems has received considerable attention along the last twenty years, see Nijmeijer and Fossen (1999) and Kreisselmeier and Engel (2003) and references therein for details. Lie algebraic methods have been employed (Krener and Isidori, 1983; Gauthier *et al.*, 1992; García and D'Attellis, 1995; Atassi and Khalil, 1999; García *et al.*, 2000; Gautier and Kupka, 2001) to transform a class of nonlinear systems into *normal forms* for which observers, with guaranteed convergence and even with nonlinear separation properties, are obtained. The Lie algebraic methods are restricted to the class of systems for which there exists a suitable state-space transformation. Smoothness is in this case instrumental in order to obtain such transformation, which may exist only locally and may be difficult to obtain. Although non-smooth systems occur frequently in practice, the results obtained are few in comparison with those for smooth systems. Among the approaches developed for non-smooth systems, that of the so-called *optimization based observer* is particularly appealing. This approach relies on the minimization of a cost functional over a moving horizon (see e.g. Zimmer, 1994 or Michalska and Mayne, 1995) and conceptually is directly linked to the observation problem, since its aim is to distinguish between different states by distinguishing their different output signals over some interval. As the idea is to store measurements from a (sliding) interval $[t-T_0, t]$, and to generate a state estimate so as to asymptotically match the predicted output with the measured one on the whole interval, this observer concept involves an infinite dimensional structure, that can at best be approximately realized at the implementation stage.

In Kreisselmeier and Engel (2003) a different observer design was presented that avoids the minimization stage of the optimization based observer, stage that under lack of smoothness and/or lack of convexity poses a tough problem. In that paper the authors introduced two concepts that characterize the variations of the output a) in terms of the difference in the initial conditions (observability) and b) as functions of time (finite complexity). These concepts are suitable for a large class of autonomous systems, which includes smooth as well as non-smooth systems. The design of the observer is based on a canonical linear model, whose dimension is the parameter to adjust, and on the construction of a partial inverse that relates the state variable of this linear model with the estimate in the original state space.

In this paper we present a generalization of the observer of Kreisselmeier and Engel (2003) to a class of controlled non-smooth nonlinear systems. With this aim, we generalize their observation and finite complexity concepts, to the case of a finite family of parameterized non-smooth systems with a unique output function. Under the hypothesis that for a suitable discretization of the control values the resulting constant-control parameterized family is observable and of finite complexity, we obtain an observer that is established in a canonical framework selecting one single parameter, the dimension of the observer, large enough. We complete the design by constructing a family of partial inverse maps, that act upon the canonical variable according to the actual value of the (sampled) control. This procedure yields a finite accuracy observer, where the observation error bound can be made arbitrarily small by increasing the dimension parameter above, and by refining the discretization of the control variable.

II. NOTATION AND PROBLEM STATEMENT

Throughout, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural integer numbers, respectively. We use $|\cdot|$

to denote the Euclidean norm on $\mathbb{R}^n$, and $|\cdot|_\infty$ to denote the supremum norm, also on $\mathbb{R}^n$. As usual, by a $\mathcal{K}$-function we mean a function $\alpha : [0, +\infty) \to [0, +\infty)$ that is strictly increasing and continuous, and satisfies $\alpha(0) = 0$. Given $U \subset \mathbb{R}^m$, we denote by $\mathcal{U}$ the set of piecewise continuous functions $u : \mathbb{R} \to U$, such that $\lim_{t\to\tau^+} u(t) = u(\tau)$, and for any $[a, b] \subset \mathbb{R}$, by $\mathcal{U}_{[a,b]}$ the restriction of the functions in $\mathcal{U}$ to $[a, b]$. By $\|\cdot\|_{[a,b]}$ we denote the $\mathcal{L}_2$- norm in the interval $[a, b]$, by $\|\cdot\|$, the $\mathcal{L}_2$-norm in $(-\infty, 0]$ and by $\langle\cdot\rangle_{[a,b]}$ the inner product of $\mathcal{L}_2$-functions on $[a, b]$.

We consider nonlinear systems of the form

$$\dot{x} = f(x, u), \qquad y = h(x) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$. We will denote with $x(\cdot, \tau, \xi, u)$ any trajectory of system (1) corresponding to the input u that verifies $x(\tau) = \xi \in \mathbb{R}^n$ and with $y(\cdot, \tau, \xi, u) = h(x(\cdot, \tau, \xi, u))$ its corresponding output. We assume that the inputs belong to the class $\mathcal{U}$, with $U \subset \mathbb{R}$ a fixed compact set. We will assume further that there exists a closed set $\mathcal{G} \subset \mathbb{R}^n$ where the state variables evolve, which is invariant with respect to (1), *i.e.*, given $\xi \in \mathcal{G}$ and $u \in \mathcal{U}$, $x(t, \tau, \xi, u) \in \mathcal{G}$ for all $t \in \mathbb{R}$.

The following assumption will be made from now on

H1: There exist positive numbers L_f and L_h such that the functions f and h verify for all $\xi, \xi' \in \mathcal{G}$ and all $\nu, \nu' \in U$

$$|f(\xi, \nu) - f(\xi', \nu')| \leq L_f(|\xi - \xi'| + |\nu - \nu'|) \tag{2}$$
$$|h(\xi) - h(\xi')| \leq L_h|\xi - \xi'|. \tag{3}$$

As stated in the introduction, our aim is to design a system

$$\dot{z} = g(z, y, u), \qquad \hat{x} = Q(z, u) \tag{4}$$

with $z \in \mathbb{R}^p$, inputs y and u from system (1) and output $\hat{x}$, which estimates the state x of system (1).

Definition II1 : Given positive numbers T and ε, we say that system (4) is a finite-time $\varepsilon - T$-observer if its solutions $z(\cdot, \tau, \zeta, u)$ are defined on $[\tau, \tau + T]$ and verify:

1. *Consistency*: Given $\xi \in \mathcal{G}$ and $u \in \mathcal{U}$, if ζ is such that $\xi = Q(\zeta, u)$, then
$$x(t, \tau, \xi, u) = \hat{x}(t, \tau, \zeta, u), \quad \forall t \in [\tau, \tau + T].$$
2. *Convergence*: For any $\zeta \in \mathbb{R}^p$, $u \in \mathcal{U}$ such that $Q(\zeta, u) \in \mathcal{G}$ and for any $\xi \in \mathcal{G}$, there exists a positive number T_u that may be made arbitrarily small such that
$$|x(t, \tau, \xi, u) - \hat{x}(t, \tau, \zeta, u)| < \varepsilon \;\; \forall t \in [\tau + T_u, \tau + T].$$

A. Observability

As stated in the Introduction we aim to generalize an existing finite-time observer, based on a concept of observability related with the output map of an autonomous system, to another, also of finite time for a controlled system. It follows that we must consider a concept of observability related with input-output maps. Consider then $T > 0$ fixed, and for each $\nu \in U$ and each $\xi \in \mathcal{G}$, let $y_\nu(\tau, \xi) = y(\tau, 0, \xi, \nu)$, $\tau \in [-T, 0]$ the output of the system seen backwards *when the constant control ν is applied.*

Definition II2 : Let $\Delta y_\nu(\tau, \xi, \xi') = y_\nu(\tau, \xi) - y_\nu(\tau, \xi')$. System (1) is *finite-time observable* if there exists $\alpha_T \in \mathcal{K}$ such that

$$\|\Delta y_\nu(\cdot, \xi, \xi')\|_{[-T,0]} \geq \alpha_T(|\xi - \xi'|), \tag{5}$$

for any $\xi, \xi' \in \mathcal{G}$ and any $\nu \in U$.

Observe that finite-time observability characterizes the variations $\Delta y_\nu(\tau, \xi, \xi')$ with respect to the distance $|\xi - \xi'|$.

Remark II3 : If $\mathcal{G}$ is compact, the observability in this sense only requires that for all $\nu \in U$, $\Delta y_\nu(\cdot, \xi, \xi') = 0$ for all $\xi = \xi' \in \mathcal{G}$.

In fact, let $d = \text{diam}(\mathcal{G})$. Due to (2)-(3) $y_\nu(\tau, \xi)$ is continuous in (ξ, ν), and hence we may take $\alpha_T \in \mathcal{K}$ defined by

$$\alpha_T(s) = \frac{s}{d} \min_{|\xi - \xi'| \geq s, \nu \in U} \|\Delta y_\nu(\cdot, \xi, \xi')\|_{[-T,0]}.$$

Definition II4 : We say that system (1) is *strongly finite-time observable* if for every $T > 0$ there exists α_T as in Definition II2 such that (5) holds .

For general nonlinear systems, this observability property will be hard to check, since observability deals with distinguishability of output solutions over $(-\infty, t]$, rather than over arbitrary small intervals. The latter is usually checked by smooth techniques, and hence is more easy to perform.

It is not hard to prove that if a systems is uniformly observable in the sense of Gauthier (see Nijmeijer and Fossen, 1999), then it is strongly finite-time observable.

Next, following Kreisselmeier and Engel (2003), we define, for each $T > 0$ and $\nu \in U$ the observation mapping

$$q_{T,\nu}(\xi) = \int_{t-T}^{t} e^{A_\nu(t-s)} b_\nu y(s, t, \xi, \nu) ds \tag{6}$$

where the pair (A_ν, b_ν) is controllable, $A_\nu \in \mathbb{R}^{p_\nu \times p_\nu}$, $b_\nu \in \mathbb{R}^{p_\nu}$, and A_ν is diagonal and Hurwitz of prescribed eigenvalues, and $|b_\nu|_\infty \leq 1$.

The mapping $q_{T,\nu} : \mathcal{G} \to \mathbb{R}^{p_\nu}$ assigns to each $\xi \in \mathcal{G}$ a point $q_{T,\nu}(\xi) \in \mathbb{R}^{p_\nu}$ via the output trajectory $y(s, t, \xi, \nu)$, $s \leq t$ of system (1).

Consider now, for each $\nu \in U$ the autonomous system defined by

$$\dot{x} = f_\nu(x) \quad y = h(x), \tag{7}$$

with $f_\nu(\cdot) = f(\cdot, \nu)$. Then, following Kreisselmeier and Engel (2003), we define for system (7) the observer

$$\begin{array}{rcl} \dot{z} & = & A_\nu z + b_\nu y(\cdot, \tau, \xi, \nu) \\ \eta(t) & = & z(t) - e^{A_\nu T} z(t-T) \\ \hat{x} & = & Q_{T,\nu}(\eta) \end{array} \tag{8}$$

defined for $t > \tau$, with initial conditions $z(t) = z_0(t), t \in [\tau - T, \tau]$ and with $Q_{T,\nu} : \mathbb{R}^{p_\nu} \to \mathbb{R}^n$, which ideally satisfies $Q_{T,\nu}(q_{T,\nu}(\xi)) = \xi$ for all $\xi \in \mathcal{G}$, and is an *extended inverse* of $q_{T,\nu}$.

We have the following result whose proof, similar to that of Theorem 5 in Kreisselmeier and Engel (2003), is included for the sake of completeness

Theorem II.5 : Suppose that

1. $q_{T,\nu} : \mathcal{G} \to \mathbb{R}^{p_\nu}$ is injective;
2. $Q_{T,\nu} : \mathbb{R}^{m_\nu} \to \mathbb{R}^n$ satisfies $Q_{T,\nu}(q_{T,\nu}(\xi)) = \xi$ for all $\xi \in \mathcal{G}$.

Then, system (8) is a finite-time observer for system (7), whose state estimate converges to the real state in finite time T, *i.e.* if $x(t, \tau, \xi, \nu)$ is a trajectory of system (7), then $\hat{x}(t) - x(t, \tau, \xi, \nu) = 0$ for all $t \geq \tau + T$.

Proof: Let us denote $x(s) = x(s, \tau, \xi, \nu)$ and

$$q_\nu(\xi) = \int_{-\infty}^{t} e^{A_\nu (t-s)} b_\nu y(s, t, \xi, \nu) ds.$$

Then, for any $t \geq \tau + T$, $q_{T,\nu}(x(t)) = q_\nu(x(t)) - e^{A_\nu T} q_\nu(x(t-T))$ and

$$\frac{d}{dt}[z(t) - q_\nu(x(t))] = A_\nu [z(t) - q_\nu(x(t))]$$

and since $\eta(t) = q_{T,\nu}(x(t)) + [z(t) - q_\nu(x(t))] - e^{A_\nu T}[z(t-T) - q_{T,\nu}(x(t-T))]$, it follows that $\eta(t) = q_{T,\nu}(x(t))$ and $\hat{x}(t) = x(t)$ for all $t \geq \tau + T$. ∎

In order a finite -time observer for system (7) to exist, it remains to establish conditions under which the hypotheses of Theorem II5 hold. With this aim, we introduce the following

Definition II.6 : Given $T > 0$ and $\nu \in U$, we say that the observation map $q_{T,\nu}$ is *uniformly injective* if there exists $\beta \in \mathcal{K}$ such that

$$|q_{T,\nu}(\xi) - q_{T,\nu}(\xi')| \geq \beta(|\xi - \xi'|)$$

for all $\xi, \xi' \in \mathcal{G}$.

The next property characterizes the variations $\Delta y_\nu(\tau, \xi, \xi')$ as functions of time τ.

Definition II.7 : Given $T > 0$, system (1) is said to be of *finite-time finite complexity* in $\mathcal{G}$ if there exists a finite number of piecewise continuous functions $\{\phi_1(\tau), \cdots, \phi_l(\tau)\}$ such that for some $\delta > 0$

$$\sum_{i=1}^{l} \left| \langle \phi_i, \Delta y_\nu(\cdot, \xi, \xi') \rangle_{[-T,0]} \right| \geq \delta \| \Delta y_\nu(\cdot, \xi, \xi') \|_{[-T,0]} \tag{9}$$

for every $\xi, \xi' \in \mathcal{G}$ and every $\nu \in U$.

Definition II.8 : We say that system (1) is of *strong finite-time finite complexity*, if for every $T > 0$ there exists δ as in Definition II.7 such that (9) holds.

Remark II.9 : The *finite-time finite complexity* and the *finite-time observability* properties assure the existence of a controllable pair (A_ν, b_ν) which renders the observation map $q_{T,\nu}$ uniformly injective in $\mathcal{G}$ (see Theorem II10 below).
On the other hand, the uniform injectivity of the map $q_{T,\nu}$ guarantees the existence of an extended inverse $Q_{T,\nu}$ for this map (Corollary II12).

We are now in position to state the following result

Theorem II.10 : Let $T > 0$ and $\nu \in U$. If system (1) is finite-time observable and of finite-time finite complexity, there exist $p_\nu \in \mathbb{N}$ and a controllable pair (A_ν, b_ν) with $A_\nu \in \mathbb{R}^{p_\nu \times p_\nu}$ Hurwitz, which can be taken diagonal and of prescribed eigenvalues, and $b_\nu \in \mathbb{R}^{p_\nu}$ with $|b_\nu|_\infty \leq 1$, such that the observation map $q_{T,\nu}$ given by (6) is uniformly injective in $\mathcal{G}$.

Proof: It follows, with minor modifications, along the line of the proof of Theorem 2 in Kreisselmeier and Engel (2003). ∎

Remark II.11 : It follows readily from Definition II.7 that if $q_{T,\nu}$ is uniformly injective for some dimension p_ν, it will also be so for every integer $m > p_\nu$. This fact will be instrumental in what follows.

From Theorem II10 and Lemma 4 in Kreisselmeier and Engel (2003), we obtain the following result

Corolary II.12 : Let $T > 0$ and $\nu \in U$. If system (1) is finite-time observable and of finite-time finite complexity, there exist an extended inverse $Q_{T,\nu}$ for $q_{T,\nu}$. Moreover $Q_{T,\nu}(\eta)$ is continuous in (η, T). □

As a consequence of this last result and of Theorem II5, the following holds.

Theorem II.13 : Let $T > 0$ and $\nu \in U$. If system (1) is finite-time observable and of finite-time finite complexity, then system (8) is a finite-time observer for system (7), whose state estimate converges to the real state in finite time T, *i.e.* if $x(t, \tau, \xi, \nu)$ is a trajectory of system (7), then $\hat{x}(t) - x(t, \tau, \xi, \nu) = 0$ for all $t \geq \tau + T$. □

In order to assure some kind of regularity on the behavior of the extended inverses, let $\Lambda = \{\lambda_i, i \in \mathbb{N}\}$ a (from now on fixed) strictly decreasing sequence of negative real numbers, and consider that $Q^{\Lambda}_{T,\nu}(\eta)$ is an extended inverse, when the eigenvalues of A_ν are the first p_ν numbers of Λ. If we denote for any $t > 0$, $Q^{\Lambda}_{\nu}(t, \eta) = Q^{\Lambda}_{t,\nu}(\eta)$, we can introduce the following

Definition II.14 : $\omega^{\Lambda,T}_{\nu}(r)$ given by

$$\omega^{\Lambda,T}_{\nu}(r) = \sup_{t\in[0,T]} \sup_{|\eta-\eta'|_\infty \le r} |Q^{\Lambda}_{\nu}(t,\eta) - Q^{\Lambda}_{\nu}(t,\eta')|$$

is a *modulus of continuity* for $Q^{\Lambda}_{T,\nu}$.

Remark II.15 : Observe that in the case that $\mathcal{G}$ is a compact set, $\omega^{\Lambda,T}_{\nu}(\cdot)$ is uniformly continuous.

Let now for each $k \in \mathbb{N}$, the set $\Lambda_k \subset \Lambda$ given by: $\Lambda_k = \{\lambda_k, \lambda_{k+1}, \cdots\}$, and consider the standing hypothesis

H2: For each $T > 0$ and each $\nu \in U$ there exists $\omega^T_\nu \in \mathcal{K}$ such that $\omega^{\Lambda_k,T}_{\nu}(r) \le \omega^T_\nu(r)$ for all $r \ge 0$ and all $k \in \mathbb{N}$.

Remark II.16 : Hypothesis **H2** states a certain kind of smoothness in the behavior of the extended inverse, considered as a function of the discrete variable p_ν, and reflects the fact that this dimension does not increase when we replace the first eigenvalues of the sequence in the determination of the controllable pair. For observable linear systems (which are of finite complexity, see Kreisselmeier and Engel (2003)), this kind of behavior is suggested by the existence of observers based on the observability Grammian (see Wonham, 1979).

We are now in position of stating the main result of this work.

Theorem II.17 : Let T' and ε positive numbers, and assume that system (1) is strongly finite-time observable and of strong finite-time finite complexity. Assume further that hypothesis **H2** holds. Then there exists an $\varepsilon - T'$ observer for (1). □

Next we obtain a series of results which will be used in the proof of this theorem.

Let $I = [a, b]$ any finite interval. We say that a finite set of real numbers $\Pi(I) = \{t_0 = a < t_1 < t_N = b\}$ is a *sampling set* for I. We say that it is a *regular sampling set of norm* μ when $t_{i+1} - t_i = \mu > 0$, $\forall i$. We denote by $\Pi(I, \mu)$ the regular sampling set of I of norm μ.

Let $\mu' > 0$, I' a compact interval and $U^* \subset U$. For $\Pi(I', \mu') = \{t_0 < t_1 < \cdots < t_N\}$, we denote by $\mathcal{PC}[\Pi(I', \mu'), U^*]$ the family of piecewise-constant, continuous from the right functions $\sigma : I' \to U^*$ such that $\sigma(t) = \sigma(t_i^+)$, $t \in [t_i, t_{i+1})$, $0 \le i < N$.

Proposition II.18 : Let ε' and T' positive numbers. Suppose that $U \subset I = [a, b]$ and let $I' = [0, T']$.Then there exists $\mu > 0$ such that for any $u \in \mathcal{U}_{I'}$ there exist $\mu_u > 0$ and $\sigma_d \in \mathcal{PC}[\Pi(I', \mu_u), \Pi(I, \mu)]$ such that

$$\int_0^{T'} |u(t) - \sigma_d(t)| dt < \varepsilon'.$$

Proof: Since u is piecewise continuous, there exist μ_u and $\sigma \in \mathcal{PC}[\Pi(I', \mu_u), U]$ such that

$$\int_0^{T'} |u(t) - \sigma(t)| dt < \frac{\varepsilon'}{2}.$$

Let μ such that $\mu T' < \varepsilon'/2$ and let $\sigma_d \in \mathcal{PC}[\Pi(I', \mu_u), \Pi(I, \mu)]$ defined by $\sigma_d(t) = u_i$ if $u_i \le \sigma(t) < u_{i+1}$, where $u_i, u_{i+1} \in \Pi(I, \mu)$. It follows readily that

$$\int_0^{T'} |\sigma_d(t) - \sigma(t)| dt < \frac{\varepsilon'}{2},$$

and in consequence, the thesis holds. ∎

Proposition II.19 : Let ε and T' positive numbers, and consider I and I' as in Proposition II.18. Then there exists $\mu > 0$ such that if $x(\cdot, 0, \xi, u)$ is the solution of (1) corresponding to $\xi \in \mathcal{G}$ and $u \in \mathcal{U}_{I'}$, then $\sigma_d \in \mathcal{PC}[\Pi(I', \mu_u), \Pi(I, \mu)]$ with $\mu_u > 0$ exists such that the solution $x_d(\cdot, 0, \xi, \sigma_d)$ of (1) verifies $|x(\tau, 0, \xi, u) - x_d(\tau, 0, \xi, \sigma_d)| < \varepsilon$ for all $\tau \in I'$.

Proof: Let us denote $x(\tau) = x(\tau, 0, \xi, u)$ and $x_d(\tau) = x_d(\tau, 0, \xi, \sigma_d)$ for the yet unknown control σ_d. Then

$$|x(\tau) - x_d(\tau)| \le \int_0^{\tau} |f(x(s), u(s)) - f(x_d(s), \sigma_d(s))| ds \le$$
$$L_f \int_0^{\tau} |x(s) - x_d(s)| ds + L_f \int_0^{T'} |u(s) - \sigma_d(s)| ds.$$

Pick ε' such that $L_f \varepsilon' \exp(L_f T') < \varepsilon$, and μ, μ_u and σ_d as in Proposition II.18. Then

$$|x(\tau) - x_d(\tau)| < L_f \int_0^{\tau} |x(s) - x_d(s)| ds + L_f \varepsilon',$$

and by Gronwall's inequality,

$$|x(\tau) - x_d(\tau)| \le L_f \varepsilon' e^{L_f \tau} < \varepsilon.$$

∎

III. THE OBSERVER

From now on we will assume that $\varepsilon > 0$ and $T' > 0$ are fixed. Let μ in Proposition II.19 corresponding to $\varepsilon/2$ and T', and suppose that $\Pi(I, \mu) = \{u_1, u_2, \ldots, u_M\}$. Assume now that we want to estimate a trajectory $x(t, 0, \xi, u)$ of (1) corresponding to $u \in \mathcal{U}_{[0,T']}$ based on the knowledge of u and of the output $y(t,0,\xi,u)$. Let then μ_u as in Proposition II.19, and $\Pi(I',\mu_u) = \{0 = t_0 < t_1 < \cdots < t_N = T'\}$. Take $T = \mu_u$ and let us define $\omega^T : [0,+\infty) \to [0,+\infty)$ by

$$\omega^T(r) = \max_{1\le k\le M} \omega^T_{u_k}(r), \quad r \ge 0.$$

Consider $\delta > 0$ such that $\omega^T(\delta) < \varepsilon/2$, and let k^* the first integer such that $|\lambda_{k^*}| > 4\varepsilon/(\delta L_h)$. Fix now the subsequence of eigenvalues $\Lambda^* = \Lambda_{k^*}$, and denote for each k, q_{T,u_k} of (6) by q_k, A_{u_k} and b_{u_k}, by A_k and b_k respectively, $Q^{\Lambda^*}_{T,u_k} = Q_k$ and $p_k = p_{u_k}$.

Proof of Theorem II.17: The following algorithm is the proposed observer

- for each k determine p_k, A_k, b_k such that an uniformly injective map q_k and its extended inverse Q_k exist. Their existence is guaranteed by Theorem II10 and Corollary II12.
- Let $p_{k^*} = \max\{p_k\}$, $A = A_{k^*}$ and $b = b_{k^*}$. Determine this time for the fixed pair (A,b) q_k and Q_k for $1 \le k \le M$. According with Remark II11, Q_k is, for each k, the extended inverse of the uniformly injective mapping q_k.
- define $Q(\cdot,u)$ by $Q(\cdot,u(t)) = Q_k(\cdot)$ if $u_k \le u(t_j) < u_{k+1}$ and $t_j \le t < t_{j+1}$
- Apply the estimator

$$\begin{array}{rcl} \dot z(t) &=& Az(t) + by(t,0,\xi,u(t)) \\ \eta(t) &=& z(t) - e^{AT}z(t-T) \\ \hat x(t) &=& Q(\eta(t),u(t)) \end{array} \qquad (10)$$

with initial condition $z(t) = z_0(t-T), 0 \le t \le T$. □

In order to prove the convergence, consider the estimator

$$\begin{array}{rcl} \dot z_d(t) &=& Az_d(t) + by_d(t,0,\xi,\sigma_d(t)) \\ \eta_d(t) &=& z_d(t) - e^{AT}z_d(t-T) \\ \hat x_d(t) &=& Q(\eta_d(t),\sigma_d(t)) \end{array} \qquad (11)$$

with $Q(\cdot,\sigma_d)$ defined as above, with initial condition $z_d(t) = z_0(t-T), 0 \le t \le T$, and with $y_d(t,0,\xi,\sigma_d(t)) = h(x_d(t,0,\xi,\sigma_d(t)))$. According to Theorem II5, $\hat x_d(t) = x_d(t)$ for every $t \in [T,T']$.

It is not hard to prove that for all $t \in [0,T']$,

$$|z(t) - z_d(t)|_\infty \le \frac{\varepsilon}{|\lambda_{k^*}|L_h} < \frac{\delta}{2}$$

and in consequence that $|\eta(t) - \eta_d(t)|_\infty \le \delta$ for all $t \in [0,T']$. It follows that for those t, $|\hat x(t) - \hat x_d(t)| = |Q(\hat x(t),u(t)) - Q(\hat x_d(t),\sigma_d(t))| < \omega^T(\delta) < \varepsilon/2$. In consequence, for every $t \in [T,T']$,

$$|x(t) - \hat x(t)| \le |x(t) - \hat x_d(t)| + |\hat x_d(t) - \hat x(t)| < \varepsilon,$$

and the theorem follows. ∎

Remark III.1 : In order to design the proposed observer we should be able to compute for each point u_k in the prescribed partition of U the injective mappings $q_k(\cdot)$ and the extended inverses Q_k. As pointed out in Kreisselmeier and Engel (2003), this can be done with an arbitrary finite accuracy, by taking the following map of approximate inversion

$$Q_k(\eta) = \frac{\int_{\mathcal{G}} \xi w_k(\epsilon,\eta,\xi)d\xi}{\int_{\mathcal{G}} w_k(\epsilon,\eta,\xi)d\xi} \qquad (12)$$

with

$$w_k(\epsilon,\eta,\xi) = \frac{1}{(\epsilon + |\eta - q_k(\xi)|)^{n+2}} \qquad (13)$$

where ϵ is chosen to achieve an accuracy of the observer high enough , (see that paper for details).

IV. AN EXAMPLE

With the purpose of exhibiting how the observer approach herein presented works, we consider the following non-autonomous Lipschitz continuous system.

$$\left\{ \begin{array}{rcl} \dot x_1 &=& x_2 u \\ \dot x_2 &=& -x_1 u \\ y &=& h(x_1) \end{array} \right. \qquad (14)$$

where $u(t)$ is a ramp that goes from 10 to 20 in 5 seconds and then descends towards 10 again in an equally large time interval.

We consider the former system of practical interest because it models the behavior of a large class of real-life devices, those consisting of a voltage controlled oscillator followed by a nonlinearity (in this case represented by $h(\cdot)$, a typical half-wave rectifier). Since for each $\nu \in U = [10,20]$ the system is piecewise linear, it is not hard to verify that the hypotheses of Corollary I of Kreisselmeier and Engel (2003) hold almost everywhere. In consequence, the strong finite-time observability and strong finite-time complexity properties are verified.

The parameters taken for the for the observer were $T' = 10$, $\varepsilon = 0.$ and $\Lambda = \{-0.01,-0.05,-0.1,-0.5,-1,-1.5,-2,-4,-8,-16\}$. The partition norm for U was $\mu = 1$, and $T = 0.5$. Via simulations, it was found that $p_{k^*} = 10$ enabled us determine the approximate inversion maps (12) - (13), implemented as

$$Q_k(\xi) = \frac{\sum_{i=1}^N x_i/[\epsilon + |q_k(x_i) - \xi|]}{\sum_{i=1}^N 1/[\epsilon + |q_k(x_i) - \xi|]}, \qquad (15)$$

with $\epsilon = 0.05$ and $\{x_i, i = 1,N\}$ a partition of $\mathcal{G} = [-1,1]\times[-1,1]$ of norm 0.0125, with an error $|Q_k(q_k(x_i)) - x_i| < 2e^{-3}$ for each k and each i.

Figures 1 to 3 show the results of the simulations, for initial conditions $x_1(0) = 1$ and $x_2(0) = 0$ and $z_i(t-0.5) = 0.2, 1 \le i \le 10, 0 \le t \le 0.5$.

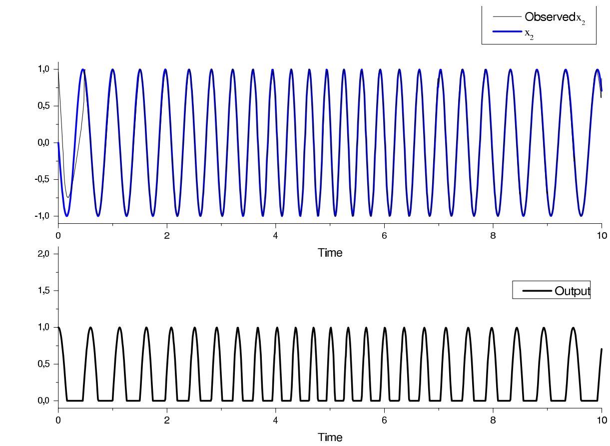

Figure 1: x_2 vs. $\hat{x}_2$ (Top) and System output (Bottom)

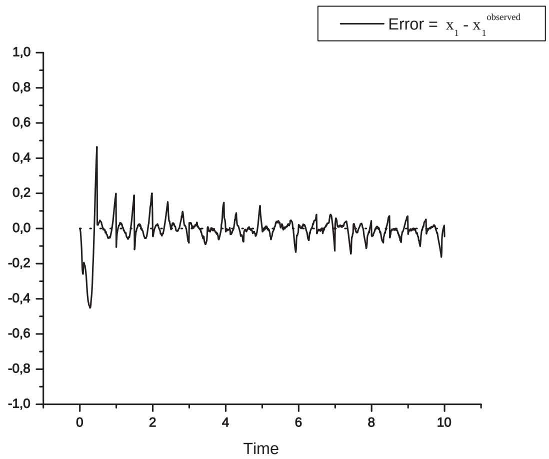

Figure 2: Estimation error for x_1

Figure 1 shows the output and the state variable x_2 and its estimation $\hat{x}_2$, while Figs. 2 and 3 show the corresponding estimation errors for x_1 and x_2 respectively. As can be seen, the observer performs within the given specifications from approximately $t = 0.5$ on. Nevertheless, in the steady state error profile there are peaks of amplitude bounded by 0.2. That happens due to the inverse map transitions that match the control switching events. This effect can be reduced at a greater computational effort by refining the control mesh and by taking T smaller.

V. CONCLUSIONS

In this paper we have presented an observer for controlled Lyapunov continuous SISO systems, (although it can be easily extended to the MIMO case), that realizes an arbitrary finite accuracy. The model of the observer is based on an existing design for autonomous systems, and applies to a rather large class of controls (that of piecewise continuous, continuous from the right controls). An example is given that exhibits the behavior of the observer.

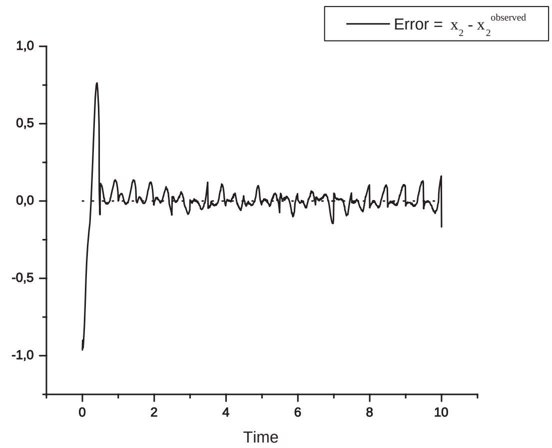

Figure 3: Estimation error for x_2

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